Bijection Between Catalan Objects

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October 3, 2018

Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

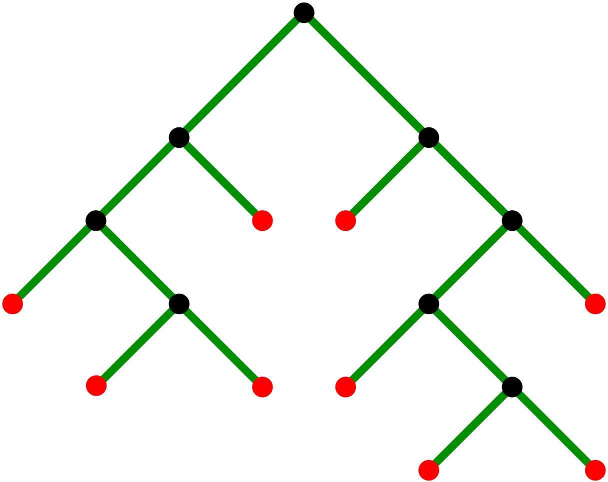
- The number of full binary tree with $2n+1$ vertices (i.e., n internal vertices).
- The number of triangulations of a convex $(n+2)$-gon.
- The number of semi-pyramid with n dimers.

There are more than 200 such objects!!

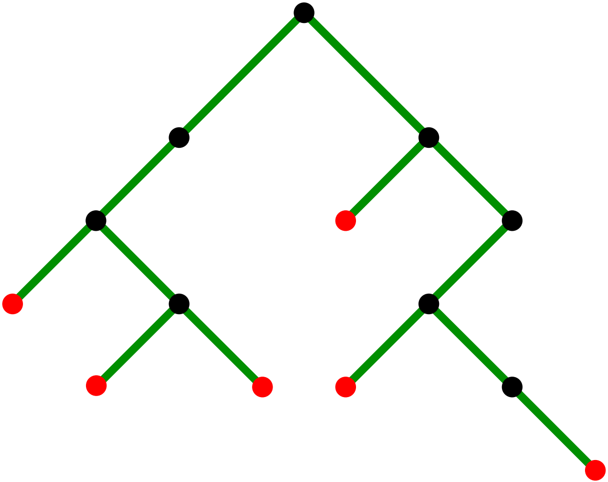
Three Types of Trees

- Binary Trees
- Full Binary Trees
- Planar Trees

Three Types of Trees



Three Types of Trees



Bijection Between Binary Trees and Full Binary Trees

Theorem

The number of binary trees (not necessarily full) of n vertices is equal to the number of full binary trees with $2n+1$ vertices.

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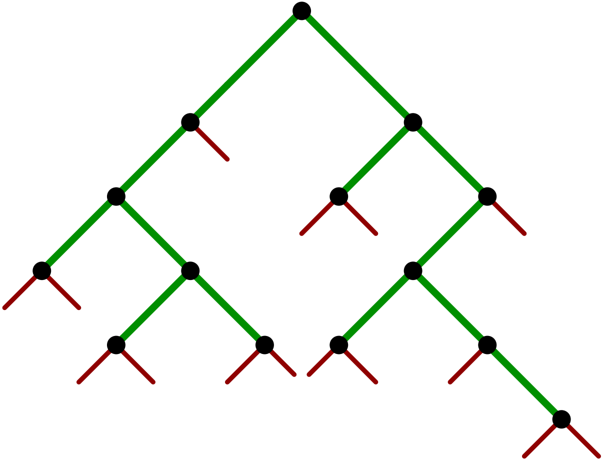
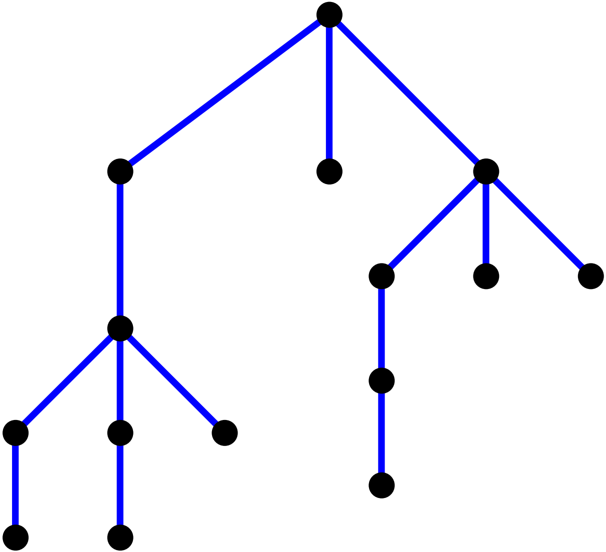
Hint: We already learned the bijection!

Bijection Between Binary Trees and Full Binary Trees



Planar Trees



The Number of Planar Trees

Theorem

The number of planar trees with $n+1$ vertices is C_n.

Exercise: Prove by generating function.

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Theorem

The number of planar trees with $n+1$ vertices is C_n.

We need to show

$$y = \sum_{n\geq 0} C_n x^{n+1} = xf$$

f is the generating function of the binary tree.

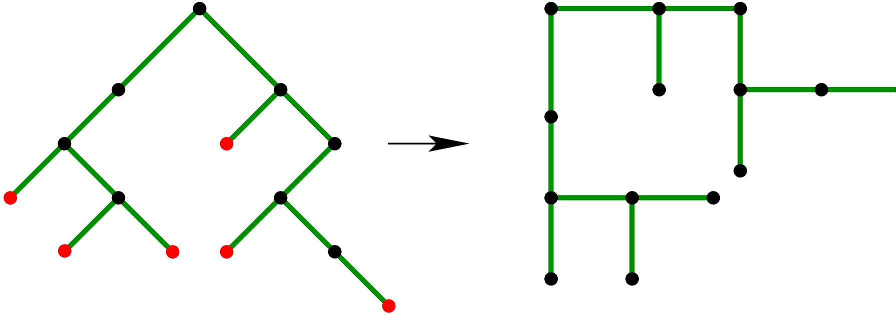
The Number of Planar Trees

Theorem

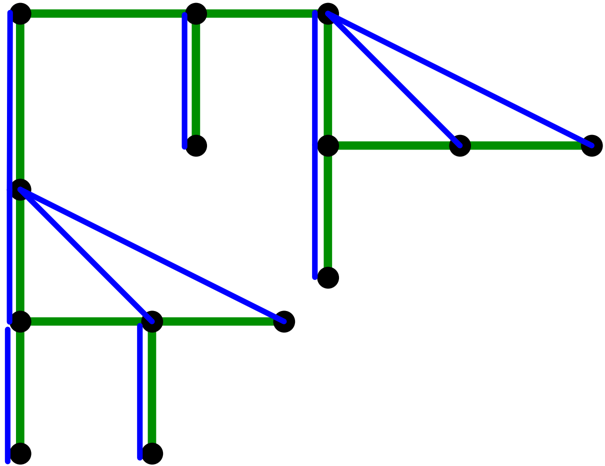
The number of planar trees with $n+1$ vertices is C_n.

- $y = x + \frac{xy}{1-y}$
- xf satisfies the same recurrence.

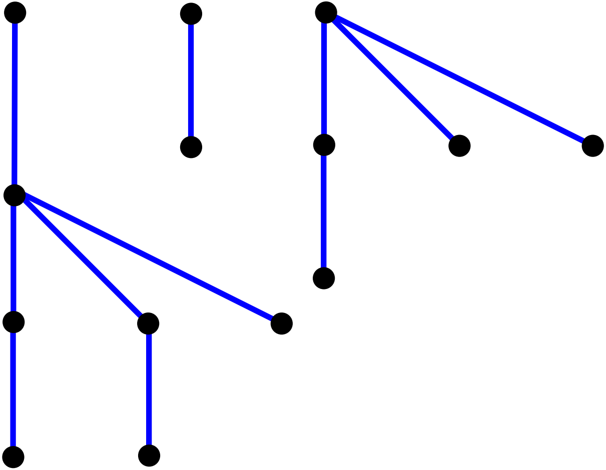
Bijection Between Binary Trees and Planar Trees

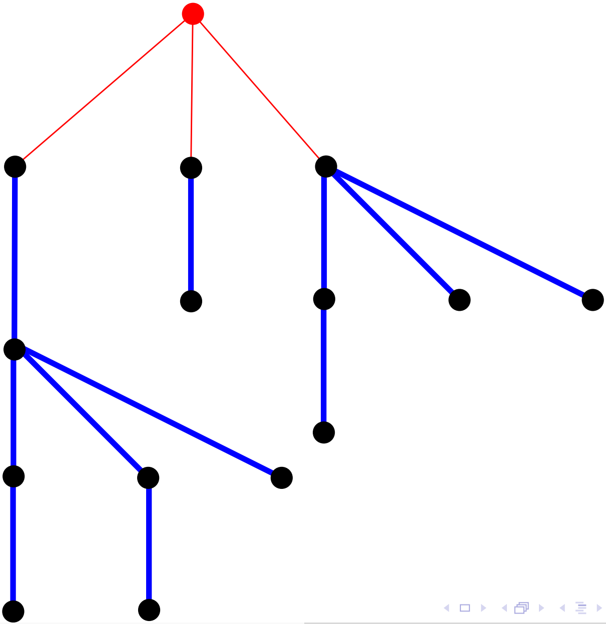


Bijection Between Binary Trees and Planar Trees



Bijection Between Binary Trees and Planar Trees





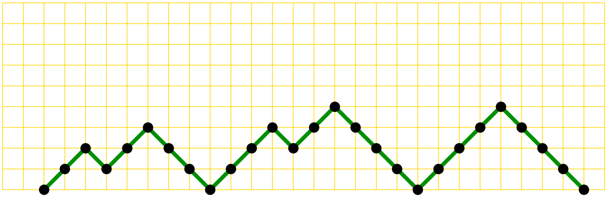
Three Types of Paths

- Dyck Paths
- 2-Colored Motzkin Paths
- Lukasiewicz Paths

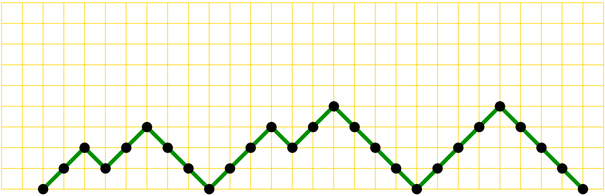
Dyck Path

A Dyck path of length $2n$ is a lattice path:

1. From $(0, 0)$ to $(2n, 0)$;
2. Use the element steps $\nearrow$ and $\searrow$;
3. Never go below the x-axis.



Dyck Path

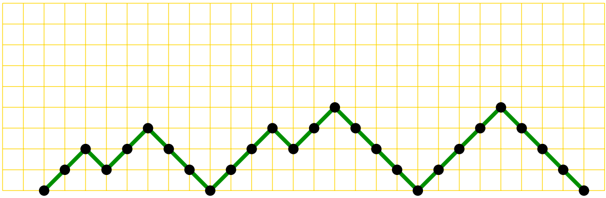


Theorem

The number of Dyck paths of length $2n$ *is* C_n.

Exercise: Prove by generating functions.

Dyck Path



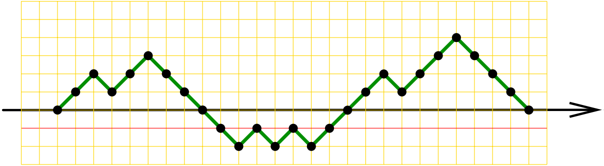
Theorem

The number of Dyck paths of length $2n$ is C_n.

Prove by reflecting principle.

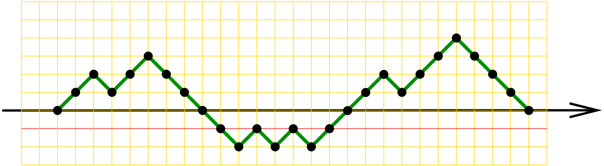
Dyck Path

\# Dyck paths = # General paths - # 'Bad' paths



Dyck Path

$$\# \text{ Dyck paths} = \binom{2n}{n} - \# \text{ 'Bad' paths}$$



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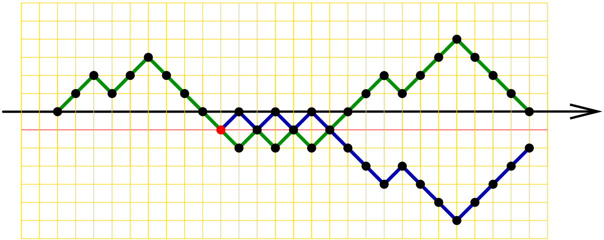

Dyck Path

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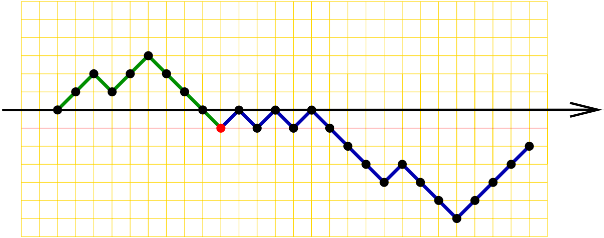
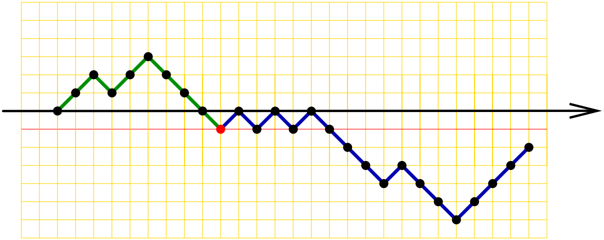


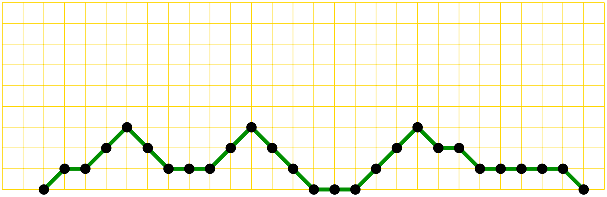
$$\# \text{ Dyck paths} = \binom{2n}{n} - \binom{2n}{n+1}$$



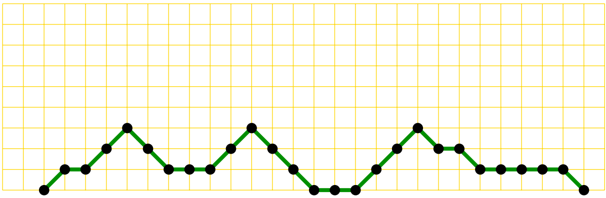
Motzkin Path

A Motzkin path of length n is a lattice path:

1. From $(0, 0)$ to $(n, 0)$;
2. Use the element steps $\nearrow$, $\searrow$, and $\rightarrow$;
3. Never go below the x-axis.



Motzkin Path



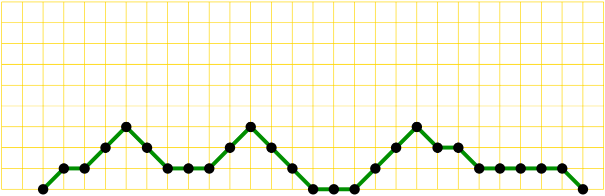
Exercise: Prove the following recurrence for the o.g.f. m of Motzkin paths:

$$m = 1 + xm + x^2 m^2$$

2-colored Motzkin Path

A 2-colored Motzkin path of length n is a lattice path:

1. From $(0, 0)$ to $(n, 0)$;
2. Use the element steps $\nearrow$, $\searrow$, and $\rightarrow$;
3. Never go below the x-axis.
4. The horizontal steps are colored by red or blue.

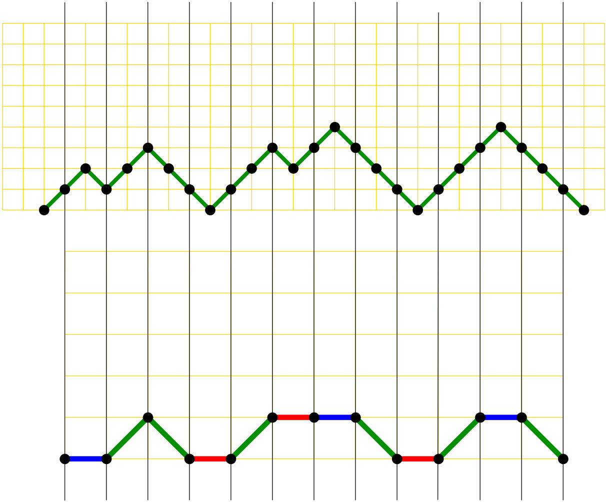


Theorem

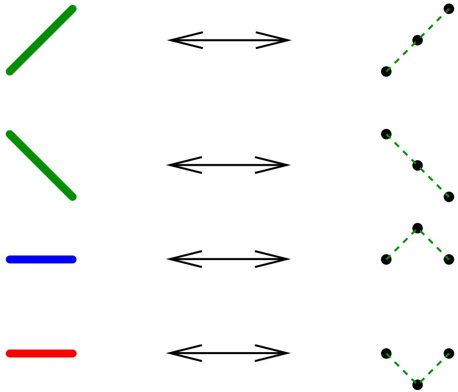
The number of 2-colored Motzkin paths of length $n-1$ is C_n.

Exercise: Prove by generating function.

Bijection: 2-colored Motzkin Paths and Dyck paths



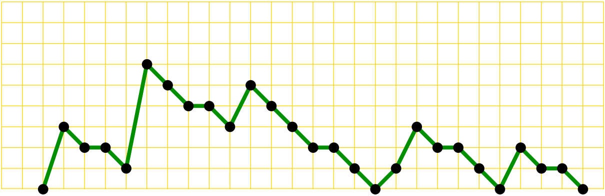
Bijection: 2-colored Motzkin Paths and Dyck paths



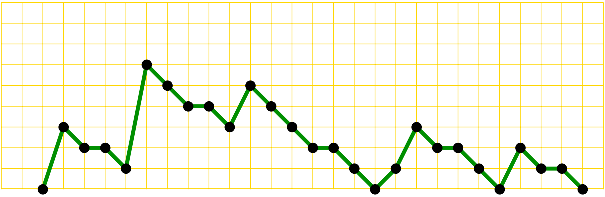
Lukasiewicz Path

A Lukasiewicz path of length n is a lattice path:

1. From $(0,0)$ to $(n,0)$;
2. Use the element steps: $\nearrow$ of arbitrary height, $\searrow$ of depth 1, and $\rightarrow$;
3. Never go below the x-axis.



Lukasiewicz Path

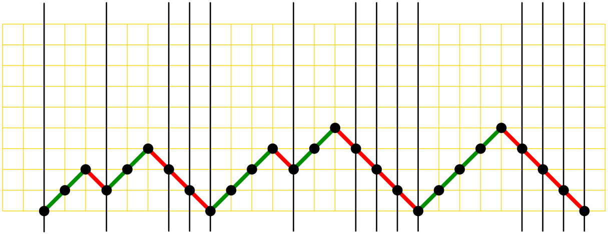


Theorem

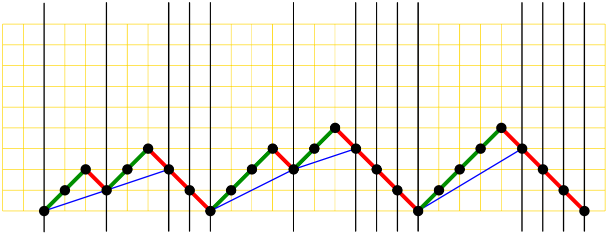
The number of Lukasiewicz paths of length n is C_n.

Prove by generating functions.

Bijection Between Dyck Paths and Lukasiewicz Paths



Bijection Between Dyck Paths and Lukasiewicz Paths



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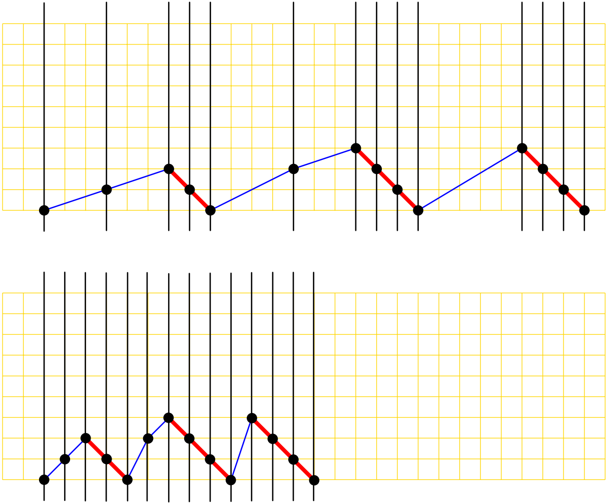


Exercise

Prove bijectively:

1. $(4n+2)C_n = (n+2)C_{n+1}$
2. (Touchard identity) $C_{n+1} = \sum_{1 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i 2^{n-2i}$

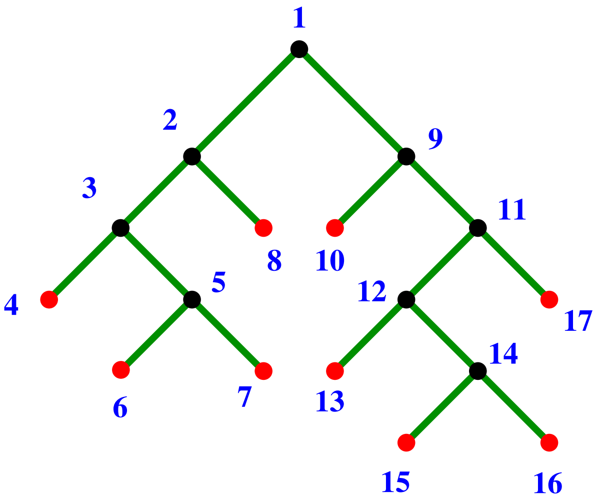
Bijection Between Trees and Paths

Left-first Search (a.k.a. 'Preoder')

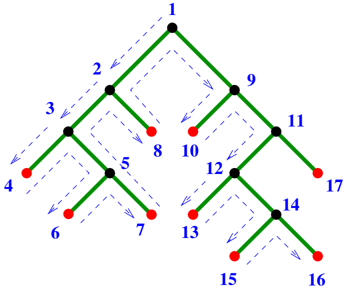
We visit the trees recursively as follow:

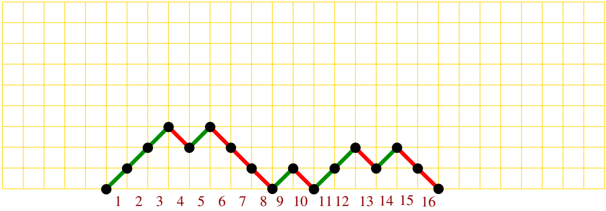
1. The root
2. The left subtree
3. The right subtree

Example



Bijection: Dyck Paths – Full Binary Tree





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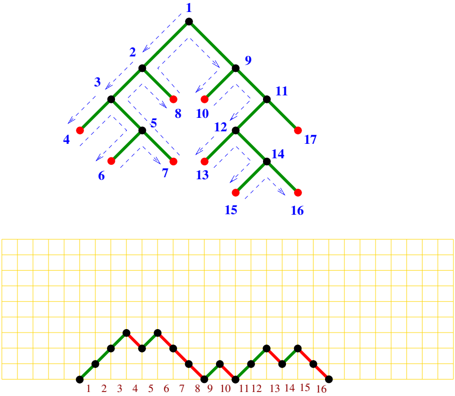
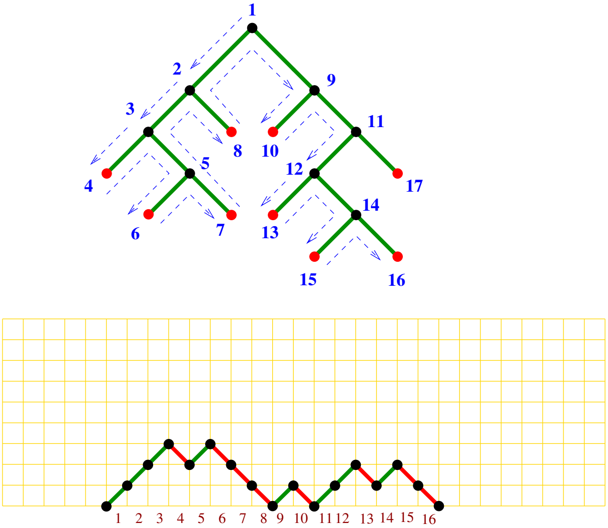


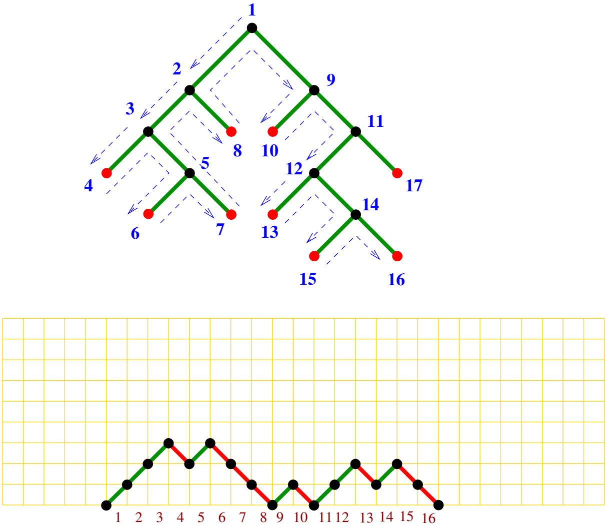
- Index the tree by the preoder;
- Travel around the tree;
- Meet an internal vertex, then go up; meet a leaf, then go down.

Bijection: Dyck Paths – Full Binary Tree



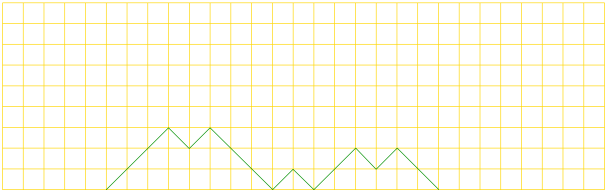
Prove that the map is well-defined.

Bijection: Dyck Paths – Full Binary Tree

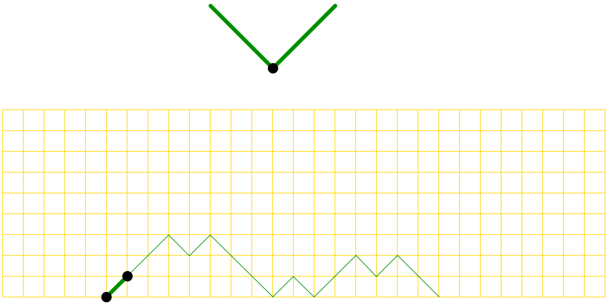


Prove that the map is indeed a bijection.

Reciprocal Bijection: Dyck Paths – Full Binary Tree



Reciprocal Bijection: Dyck Paths – Full Binary Tree



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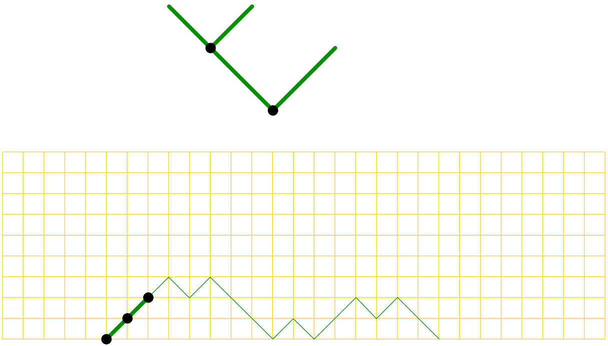

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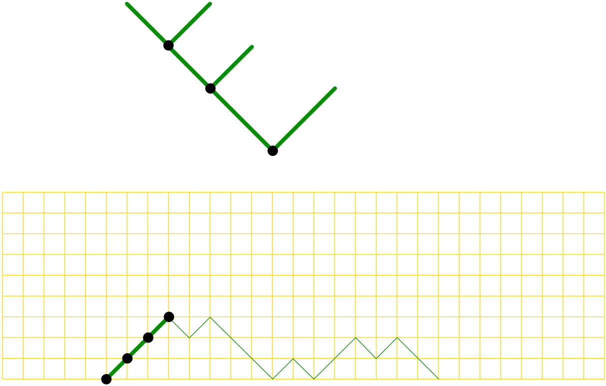

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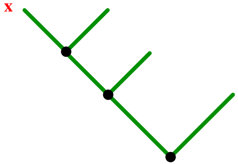

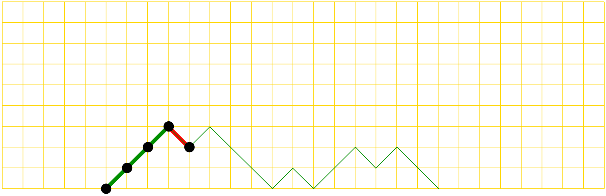

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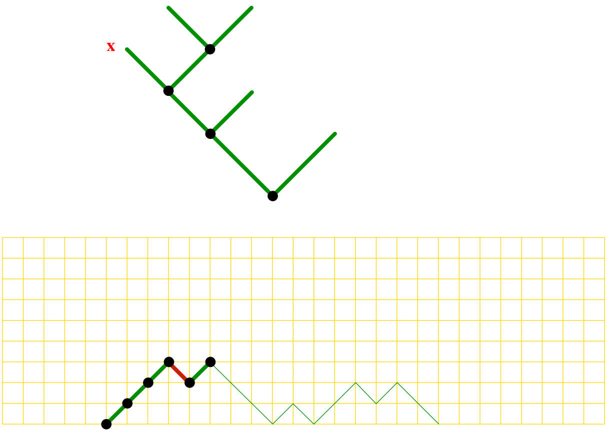

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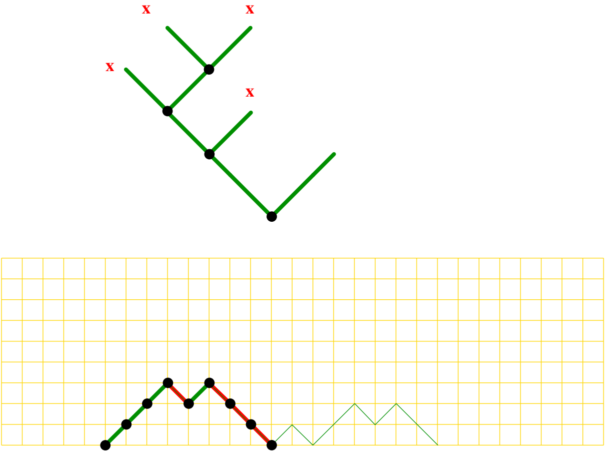
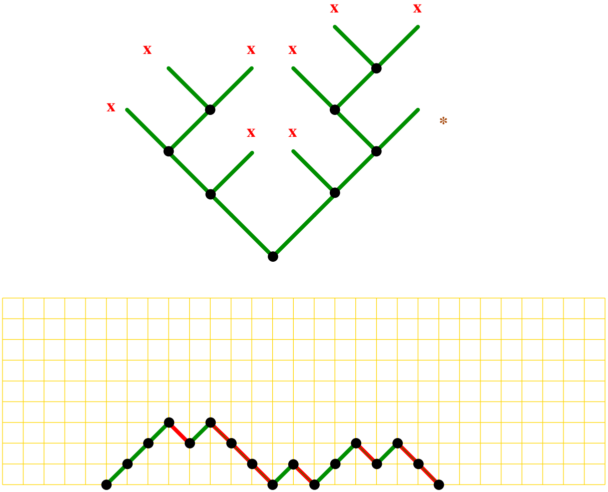
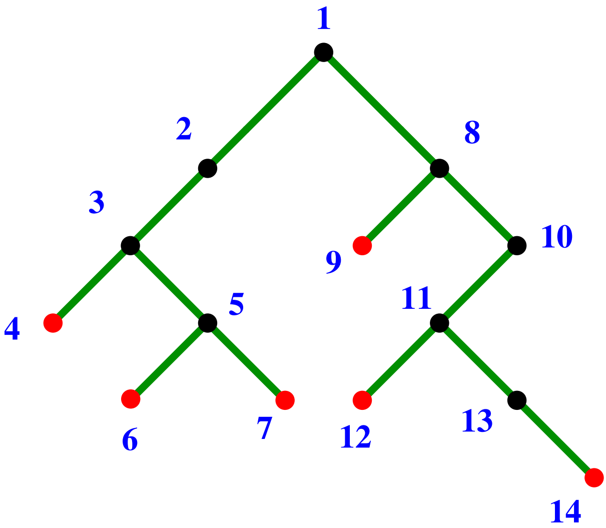


Reciprocal Bijection: Dyck Paths – Full Binary Tree



Bijection: Binary Trees and Motzkin Paths



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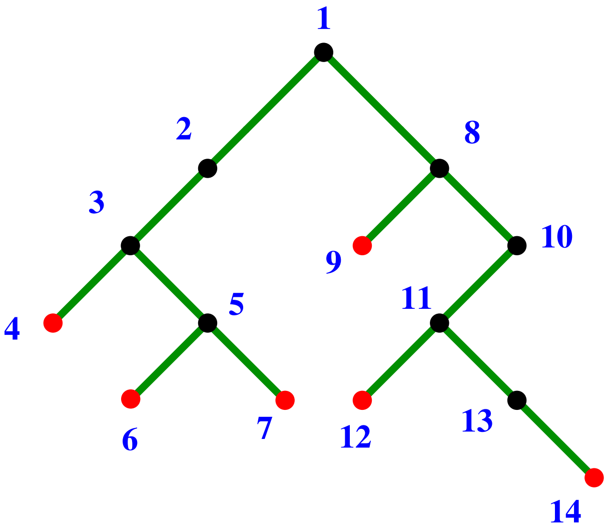
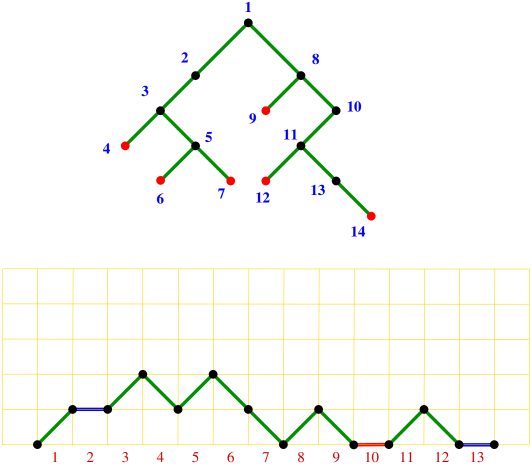
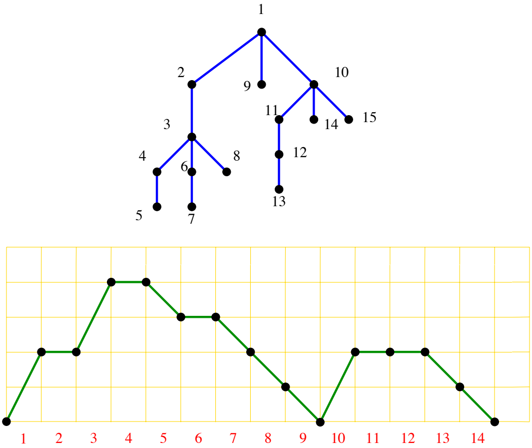


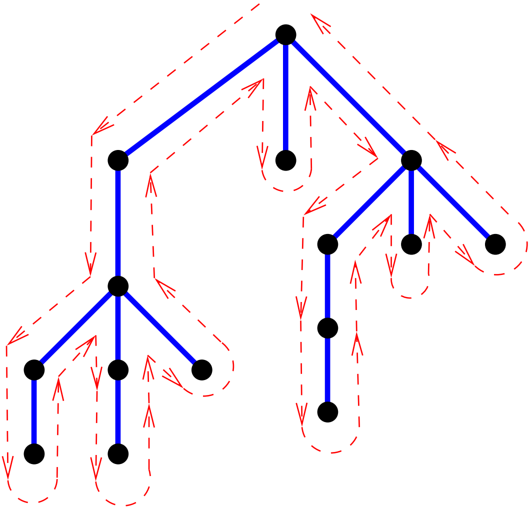


Exercise: Prove that the map is well-defined and that the map is bijective.



The height of step i is $d(i) - 1$.

Bijection: Planar Trees and Dyck Paths



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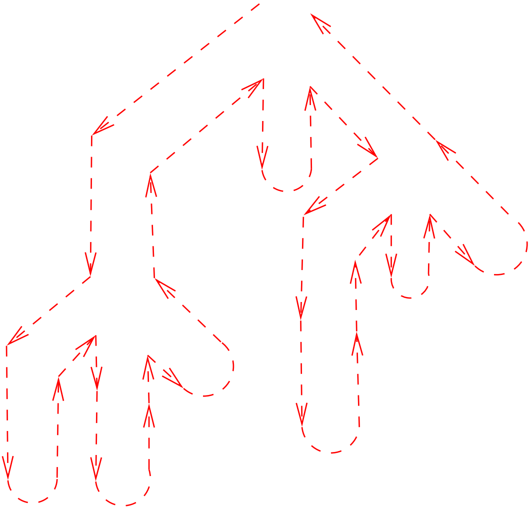

Bijection: Planar Trees and Dyck Paths

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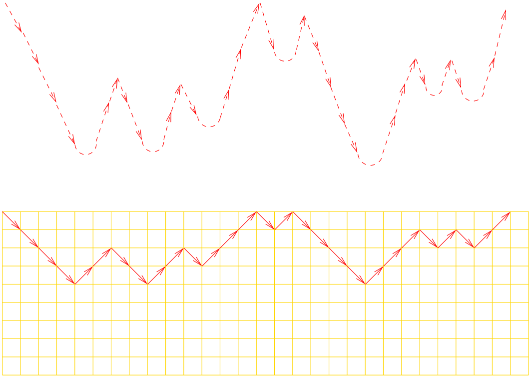

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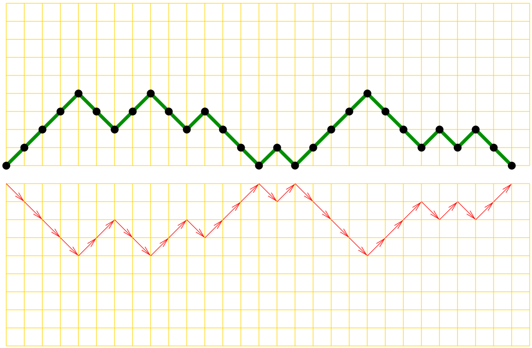
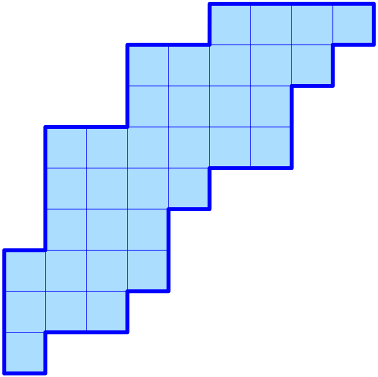
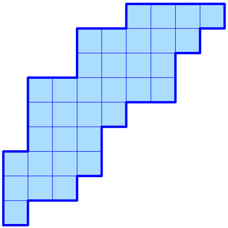


Bijection on Staircase Polygons



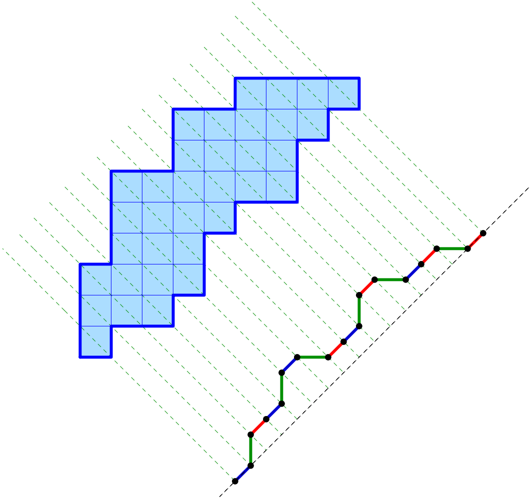
The Number of Staircase Polygons



Theorem

The number of staircase polygons of perimeters $2n+2$ is equal to C_n.

Bijection: Staircase Polygons – 2-colored Motzkin Paths



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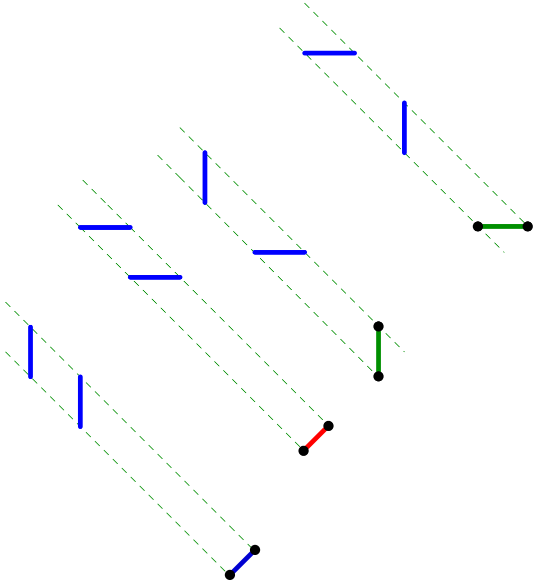

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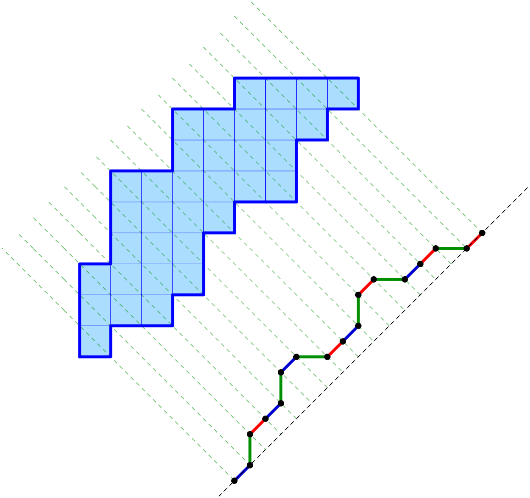
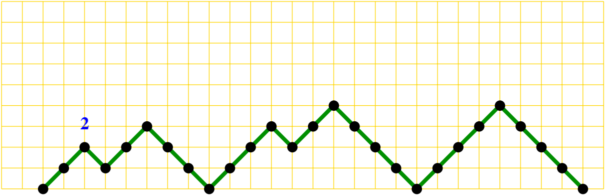


Bijection: Staircase Polygons – Dyck Paths



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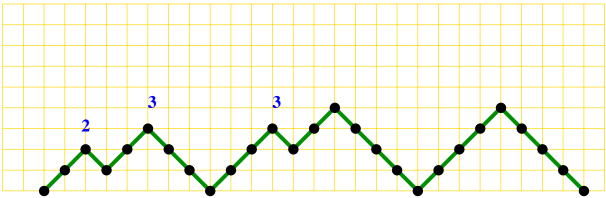

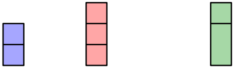

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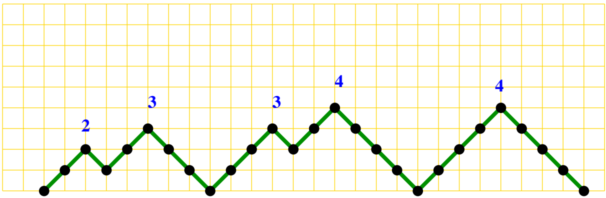
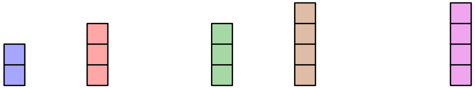
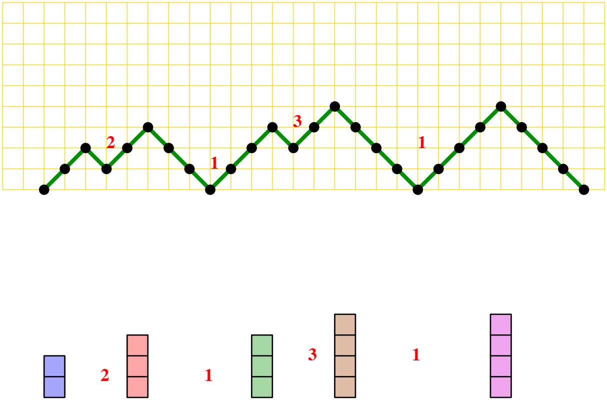
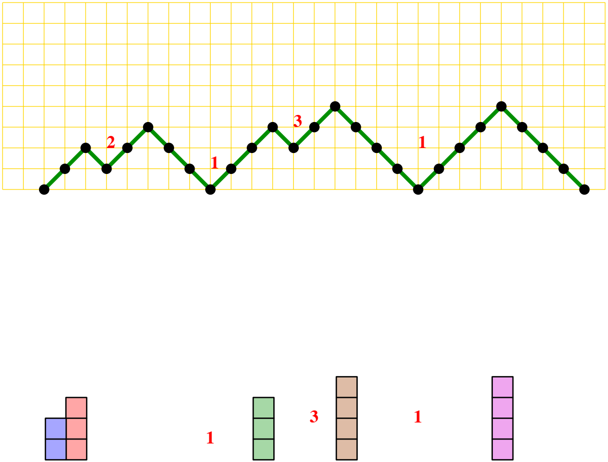




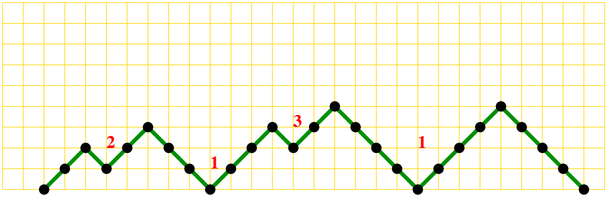
Bijection: Staircase Polygons – Dyck Paths



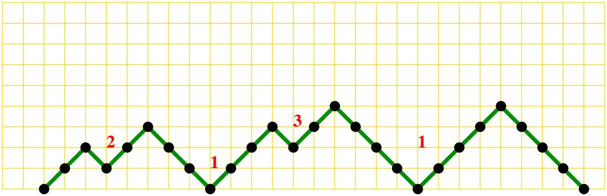
Bijection: Staircase Polygons – Dyck Paths



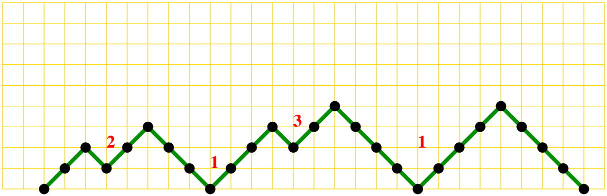
Bijection: Staircase Polygons – Dyck Paths

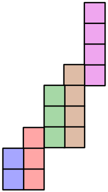


Bijection: Staircase Polygons – Dyck Paths



Bijection: Staircase Polygons – Dyck Paths



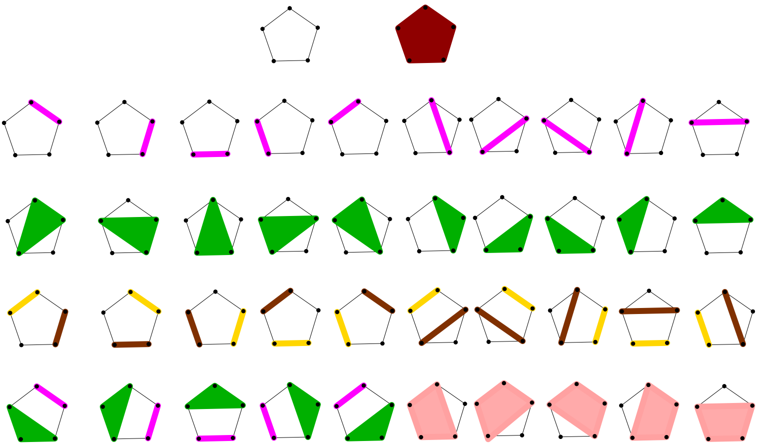


A non-crossing partition of $\{1, 2, \ldots, n\}$ is a set partition $\{B_1, B_2, \ldots, B_k\}$ such that: There are no $a < b < c < d$ with $a, c \in B_i$ and $b, d \in B_j$ ($i \neq j$).

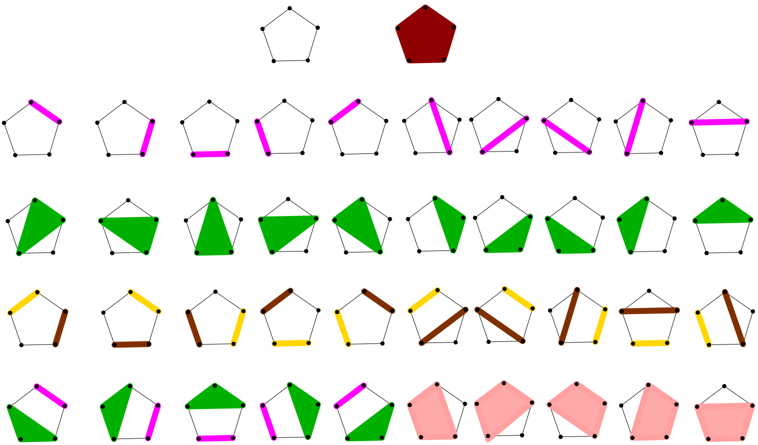
Non-crossing partitions

- Visualize the n-set S as the vertex set of a regular n-gon.
- Each subset (or block) in a set partition of S is a polygon containing the corresponding vertices.
- A non-crossing partition is a set partitions such that the polygons corresponding to the blocks are non-intersecting.

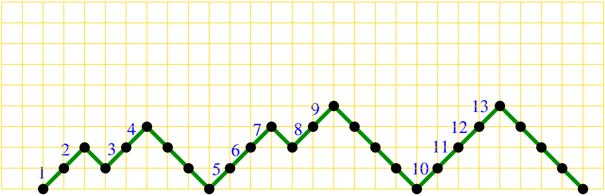
Non-crossing partitions



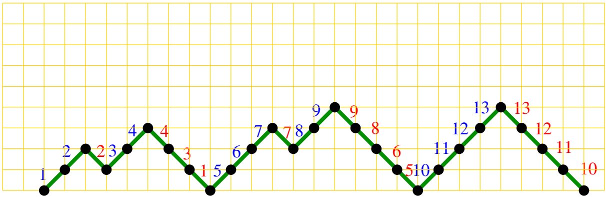
Non-crossing partitions



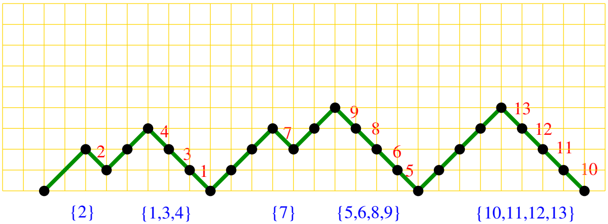
Bijection: Non-crossing partitions– Dyck Paths



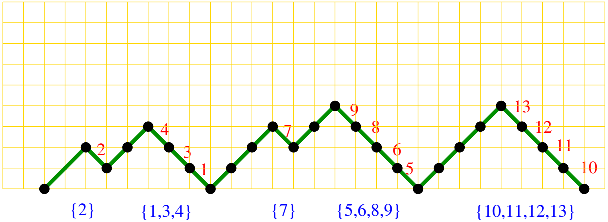
Bijection: Non-crossing partitions– Dyck Paths

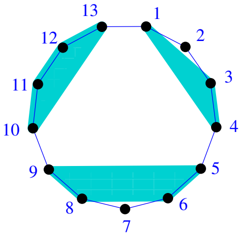


Bijection: Non-crossing partitions– Dyck Paths



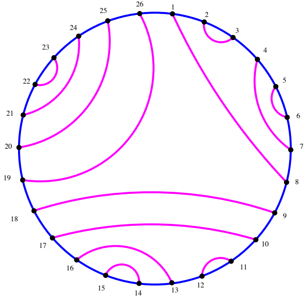
Bijection: Non-crossing partitions– Dyck Paths





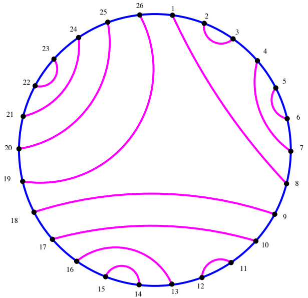
Homework: Prove that this map is well-defined and is a bijection.

Chord Diagrams



- Paring $2n$ vertices around the circle by chords;
- The chords are non-intersecting.

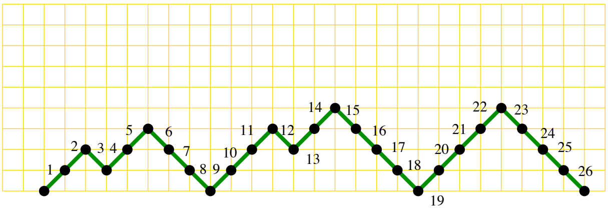
Chord Diagrams



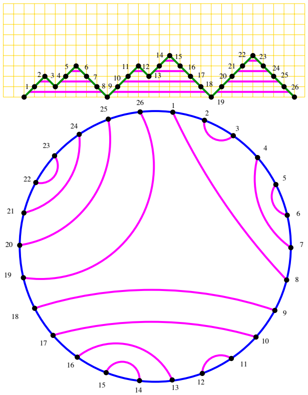
Theorem

The number of chord diagrams of $2n$ vertices is C_n.

Bijection: Dyck Paths – Chord Diagrams



Bijection: Dyck Paths – Chord Diagrams



Parenthesis System

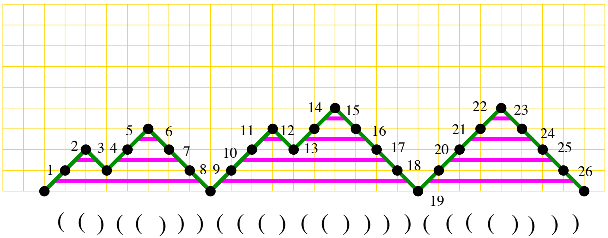
- Legal: ((()())())(()()), ((())()(()()))()()
- Ilegal: (()(())))()()((), ((()(()()))))(()()(()))

Theorem

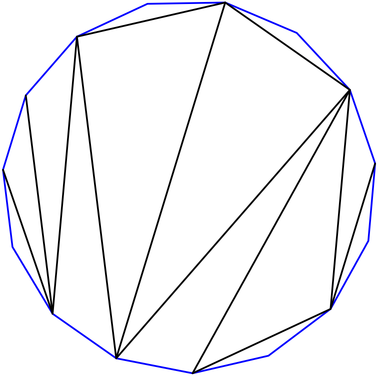
The number of (legal) systems of n pairs of parentheses is C_n.

Bijection: Dyck Paths –Parenthesis System

- Legal: ((()())())(()()), ((())()(()()))()()
- Ilegal: (()(()))$\color{red}{)}$()()((), ((()(()())))$\color{red}{)}$(()()(()))



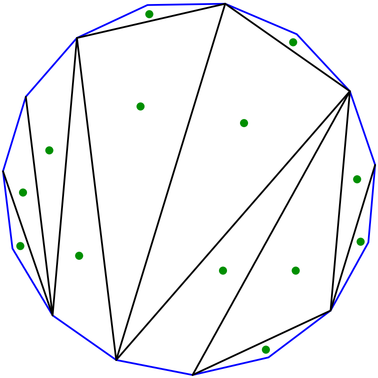
Triangulation of convex polygon



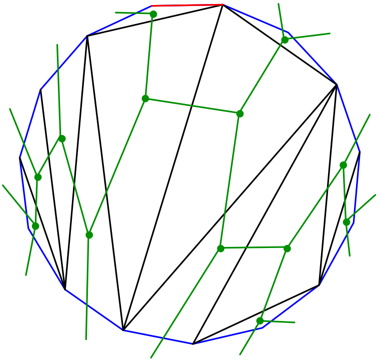
Theorem

The number of triangulations of a convex $(n+2)$-gon is C_n.

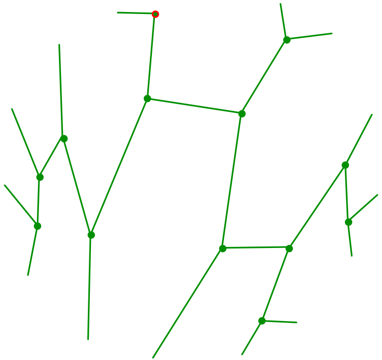
Bijection: Triangulations – Full Binary Trees



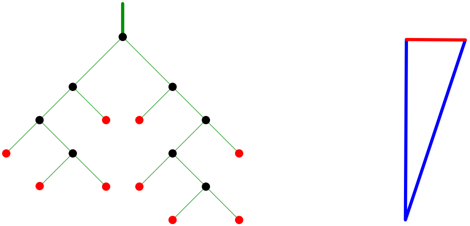
Bijection: Triangulations – Full Binary Trees



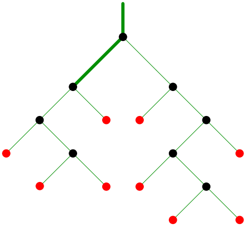
Bijection: Triangulations – Full Binary Trees

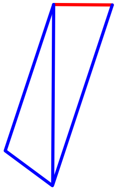


Reciprocal Bijection: Triangulations – Full Binary Trees

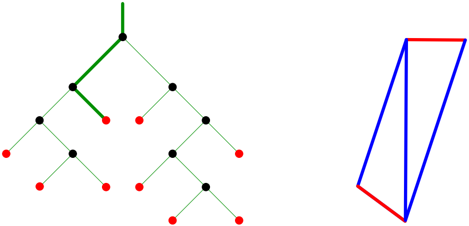


Reciprocal Bijection: Triangulations – Full Binary Trees

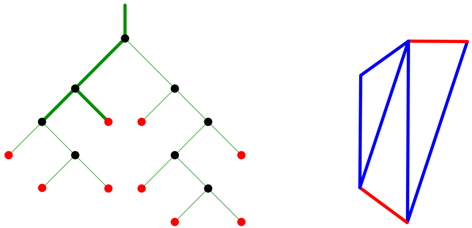




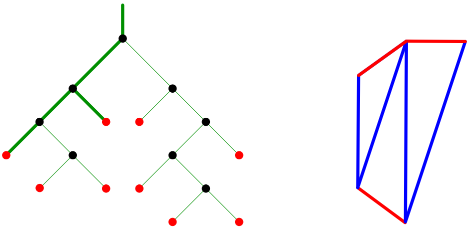
Reciprocal Bijection: Triangulations – Full Binary Trees



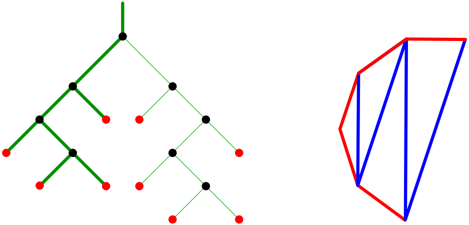
Reciprocal Bijection: Triangulations – Full Binary Trees



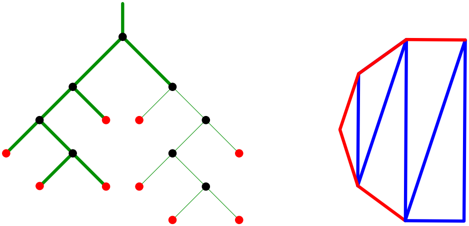
Reciprocal Bijection: Triangulations – Full Binary Trees



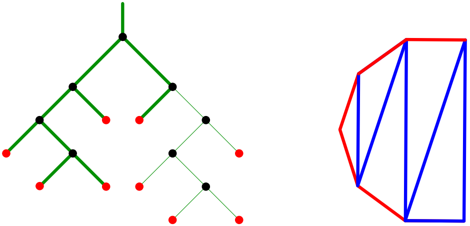
Reciprocal Bijection: Triangulations – Full Binary Trees



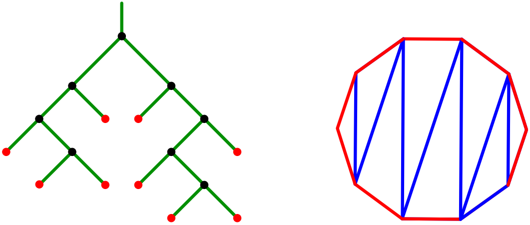
Reciprocal Bijection: Triangulations – Full Binary Trees



Reciprocal Bijection: Triangulations – Full Binary Trees



Reciprocal Bijection: Triangulations – Full Binary Trees



Ordered Pair of Increasing Sequences

A pair of increasing sequences $0 < a_1 < a_2 < \ldots < a_k < n$ and $0 < b_1 < b_2 < \ldots < b_k < n$ is said to be ordered if $a_i \geq b_i$.

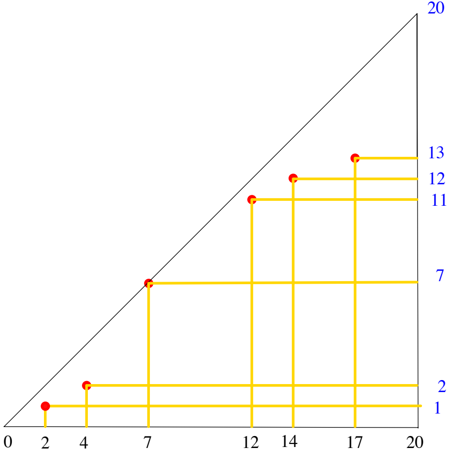
Example:

2	4	7	12	14	17
1	2	7	11	12	13

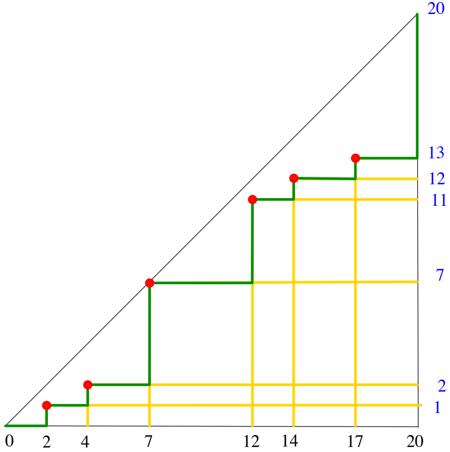
Theorem

The number of ordered pairs of sequence of order n is C_n.

Bijection: Ordered Pairs –Dyck Paths



Bijection: Ordered Pairs –Dyck Paths

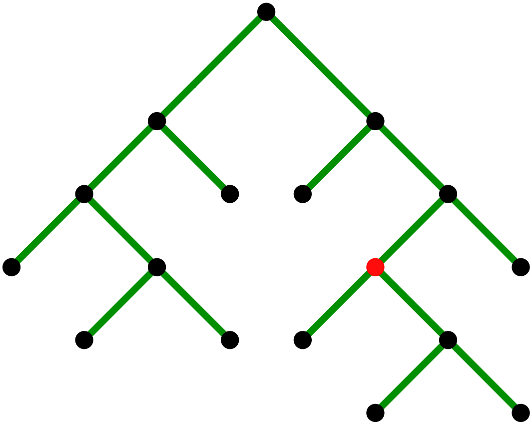


Exercise

Prove bijectively:

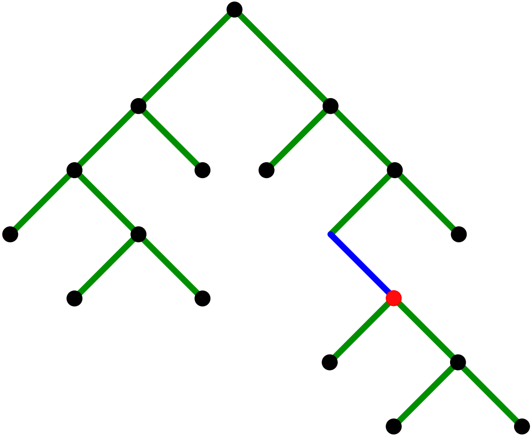
1. $(4n+2)C_n = (n+2)C_{n+1}$
2. (Touchard identity) $C_{n+1} = \sum_{1 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i 2^{n-2i}$

Prove $(4n+2)C_n = (n+2)C_{n+1}$



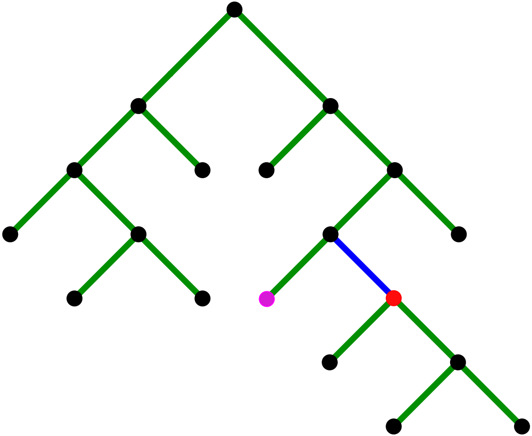
Pick a vertex randomly from a full binary tree with $2n+1$ vertices.

Prove $(4n+2)C_n = (n+2)C_{n+1}$



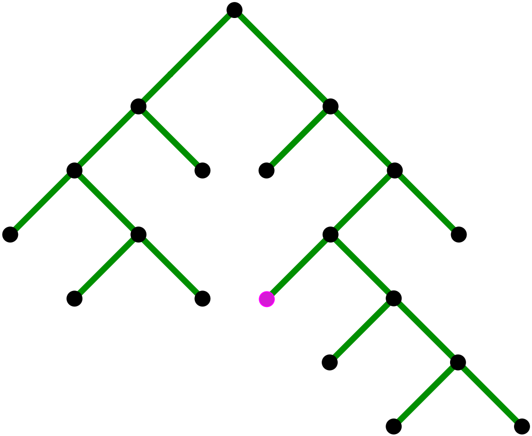
Choose one of two options: slide the subtree to the left or to the right.

Prove $(4n+2)C_n = (n+2)C_{n+1}$



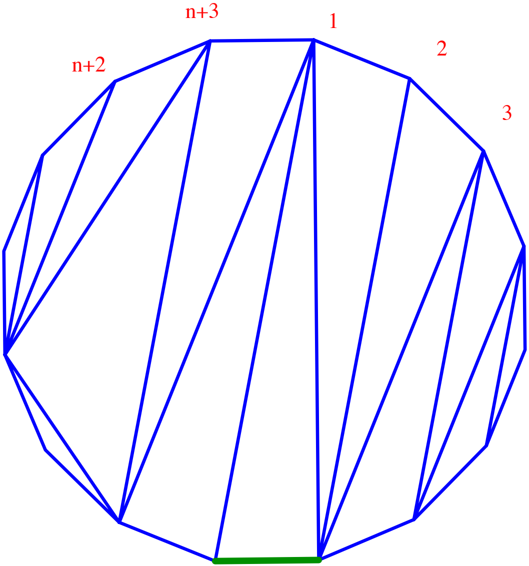
Add the missing leaf, and mark it.

Prove $(4n+2)C_n = (n+2)C_{n+1}$



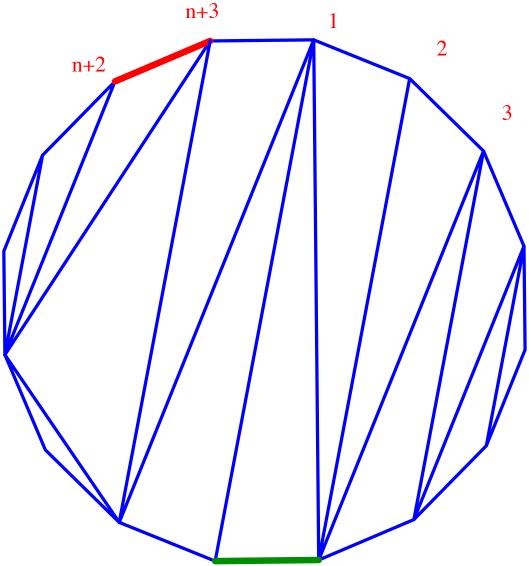
We get a full binary tree with $2n+3$ vertices and a marked leaf.

Prove $(4n+2)C_n = (n+2)C_{n+1}$: Second solution.



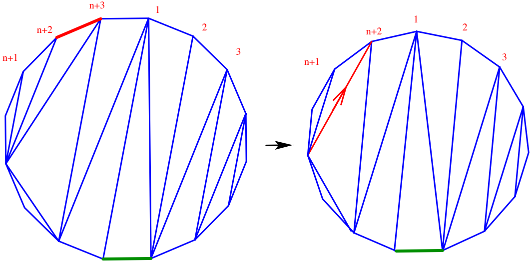
Pick a base in a $(n+3)$-gon.

Prove $(4n+2)C_n = (n+2)C_{n+1}$: Second solution.



Pick a base in a $(n+3)$-gon, then mark randomly an edge of the polygon

Prove $(4n+2)C_n = (n+2)C_{n+1}$: Second solution.



Collapse the marked edge to obtain a triangulation of a $(n+2)$-gon, marked and orient the merged edge or diagonal.

Exercise

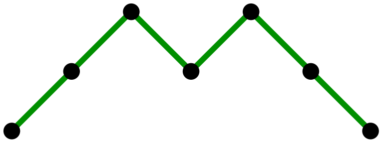
Prove bijectively:

1. $(4n+2)C_n = (n+2)C_{n+1}$
2. (Touchard identity) $C_{n+1} = \sum_{1 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i 2^{n-2i}$

Prove Touchard Identity

Work on the RHS:

- Pick a Dyck path of length $2i$ in C_i ways.



Prove Touchard Identity

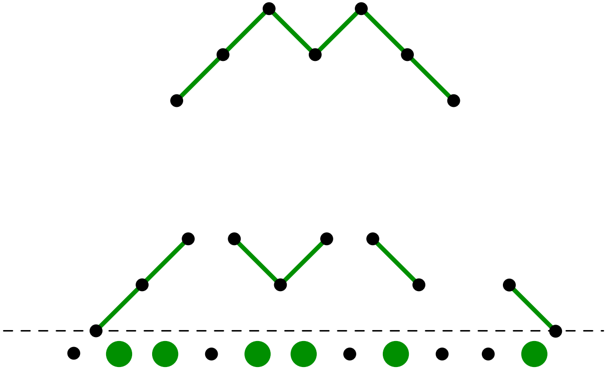
- Pick a $2i$-subset of $\{1, 2, \ldots, n\}$ in $\binom{n}{2i}$ ways.

Prove Touchard Identity

- Pick a $2i$-subset of $\{1, 2, \ldots, n\}$ in $\binom{n}{2i}$ ways.

Work on the RHS:

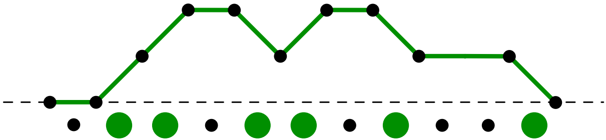
- Break the Dyck path.



Prove Touchard Identity

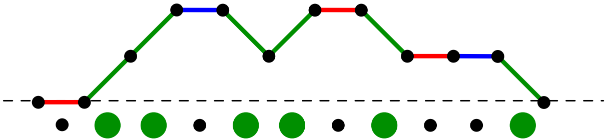
Work on the RHS:

- Add $n - 2i$ horizontal steps.



Work on the RHS:

- Color $n - 2i$ horizontal steps in 2^{n-2i} ways.



Prove Touchard Identity

Work on the RHS:

- Pick a Dyck path of length $2i$ in C_i ways.
- Pick a $2i$-subset of $\{1, 2, \ldots, n\}$ in $\binom{n}{2i}$ ways.
- Break the Dyck path and add $n - 2i$ horizontal steps.
- Color $n - 2i$ horizontal steps in 2^{n-2i} ways.
- Get a 2-colored Motzkin paths of length n.

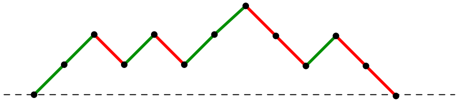
Corollary

The number of 2-colored Motzkin paths of length n with $n - 2i$ flat steps is $\binom{n}{2i} C_i 2^{n-2i}$.

Restricted Dyck paths

Theorem

The number of UUU-free Dyck paths of length $2n$ is M_n, the number of (monochromatic) Motzkin paths of length n.



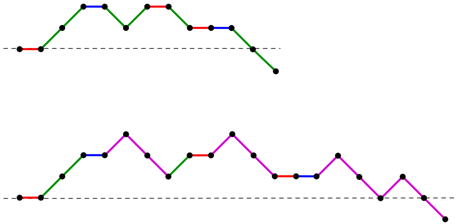
Exercise: Prove the theorem.

Restricted Dyck paths

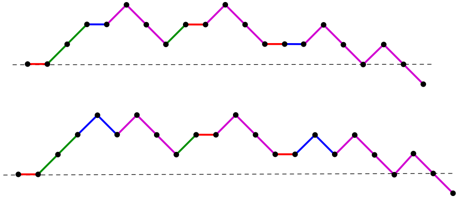
Theorem

(i) The number of Dyck paths of length $2n$ containing exactly k 'UDU's is $\binom{n-1}{k} M_{n-1-k}$.
(ii) The number of Dyck paths of length $2n$ containing exactly k 'DDU's is $\binom{n-1}{2k} 2^{n-1-2k} C_k$.

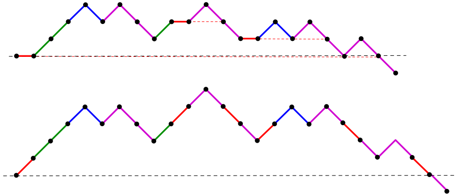
Bijection: 2-colored Motzkin paths – Restricted Dyck paths (Callan 2004)



Append a D step. Replace D by UDD.

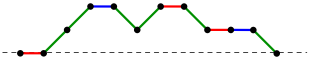


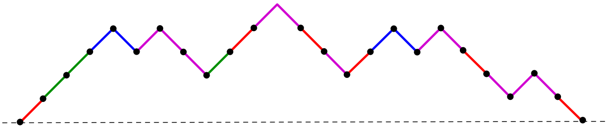
Replace F by UD.



Replace *F* by *U* and insert a *D* immediately before its associated down step. Remove the last *D* step.

Bijection: 2-colored Motzkin paths – Restricted Dyck paths





Exercise:

1. Prove that this map is indeed a bijection.
2. # flat steps in 2-colored Motzkin path = # UDUs in the Dyck path.
3. # down steps in 2-colored Motzkin path = # DDUs in the Dyck path.

Proof of part (i)

- # Dyck paths of length $2n$ with k UDUs= # 2-colored Motzkin paths of length $n-1$ with k flat steps.

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- # Dyck paths of length $2n$ with k UDUs= # 2-colored Motzkin paths of length $n-1$ with k flat steps.
- Cut off these k flat steps from the Motzkin path, we get a monochromatic Motzkin path of length $n-1-k$.

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- # 2-colored Motzkin paths of length $n-1$ with k flat steps= $\binom{n-1}{k} \times$ # monochromatic Motzkin path of length $n-1-k$.
- # Dyck paths of length $2n$ with k UDUs= $\binom{n-1}{k} \times$ # monochromatic Motzkin path of length $n-1-k$

Proof of part (ii)

- $\#$ Dyck paths of length $2n$ with k DDUs $= \#$ 2-colored Motzkin paths of length $n-1$ with k down steps.

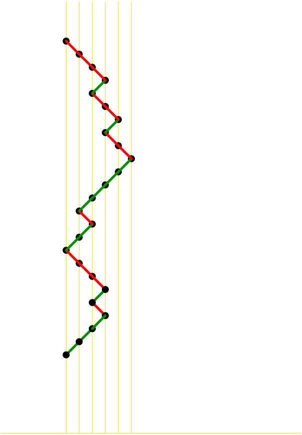
Proof of part (ii)

- # Dyck paths of length $2n$ with k DDUs = # 2-colored Motzkin paths of length $n-1$ with k down steps.
- (Corollary in proof of Touchard Identity) # 2-colored Motzkin paths of length $n-1$ with k down steps$=\binom{n-1}{2k}2^{n-1-2k}C_k$.

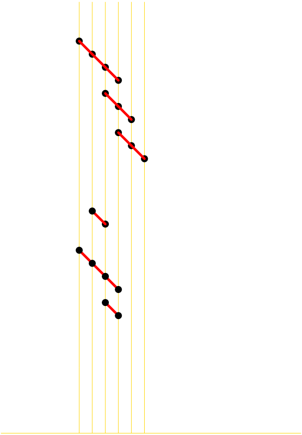
Proof of part (ii)

- # Dyck paths of length $2n$ with k DDUs = # 2-colored Motzkin paths of length $n-1$ with k down steps.
- (Corollary in proof of Touchard Identity) # 2-colored Motzkin paths of length $n-1$ with k down steps$=\binom{n-1}{2k}2^{n-1-2k}C_k$.
- # Dyck paths of length $2n$ with k DDUs $=\binom{n-1}{2k}2^{n-1-2k}C_k$.

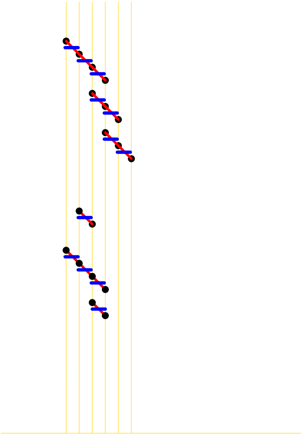
Bijection: Dyck paths – Semi-pyramids



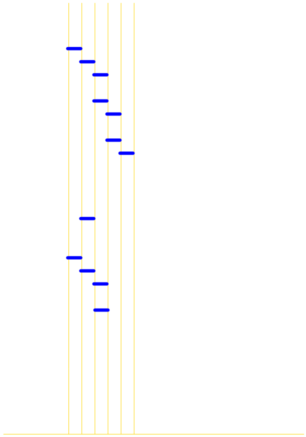
Bijection: Dyck paths – Semi-pyramids



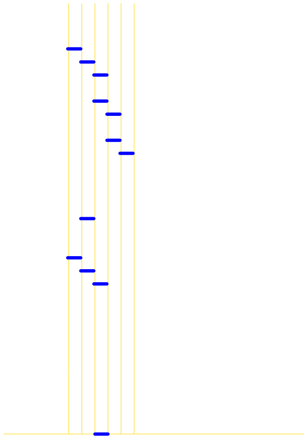
Bijection: Dyck paths – Semi-pyramids



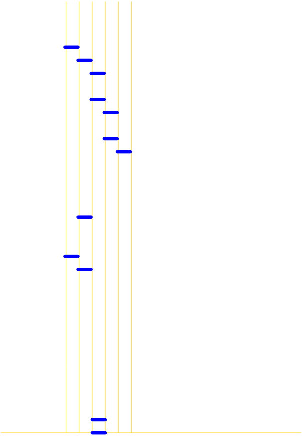
Bijection: Dyck paths – Semi-pyramids



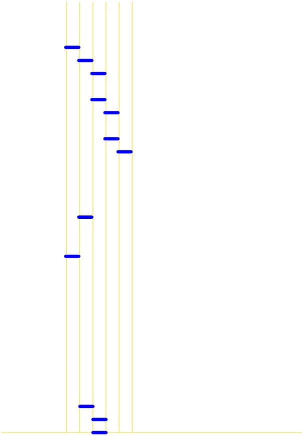
Bijection: Dyck paths – Semi-pyramids



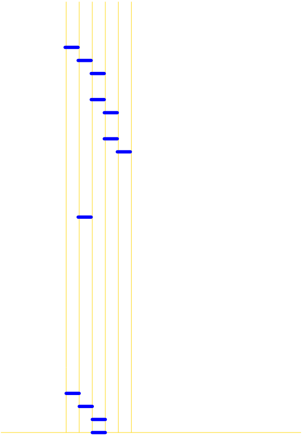
Bijection: Dyck paths – Semi-pyramids



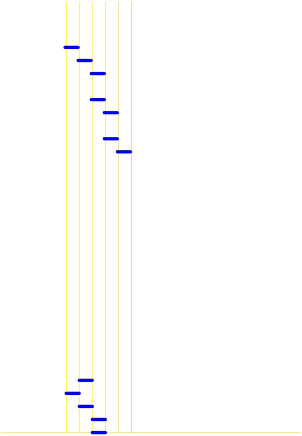
Bijection: Dyck paths – Semi-pyramids



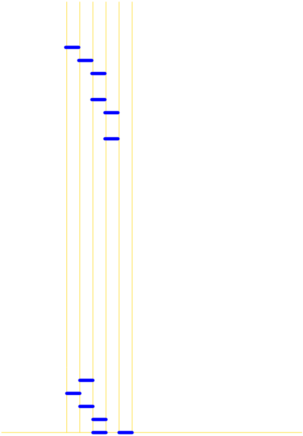
Bijection: Dyck paths – Semi-pyramids



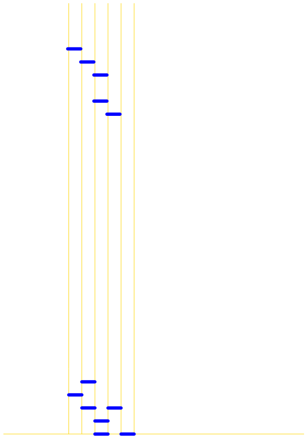
Bijection: Dyck paths – Semi-pyramids



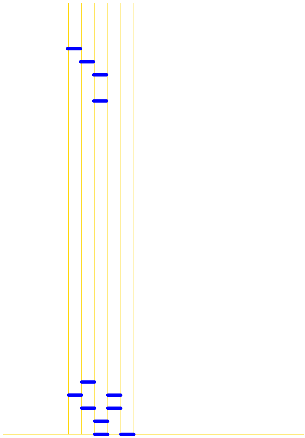
Bijection: Dyck paths – Semi-pyramids



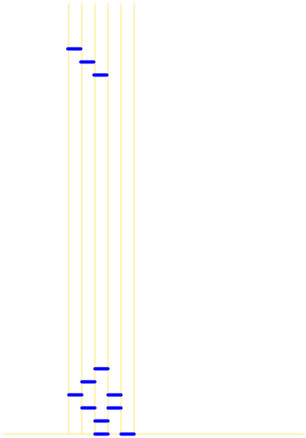
Bijection: Dyck paths – Semi-pyramids



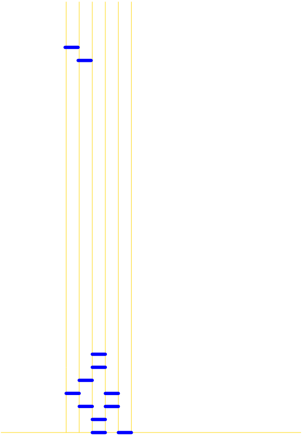
Bijection: Dyck paths – Semi-pyramids



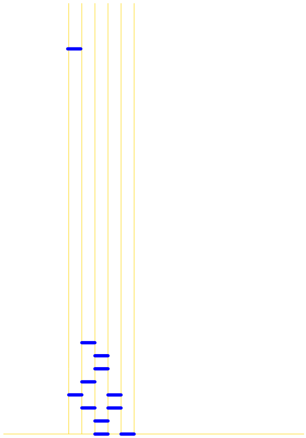
Bijection: Dyck paths – Semi-pyramids



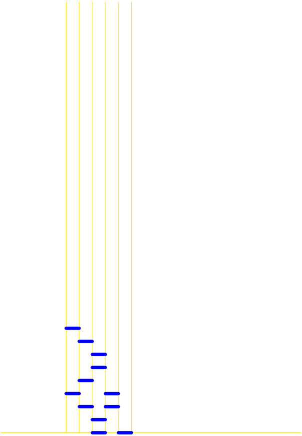
Bijection: Dyck paths – Semi-pyramids



Bijection: Dyck paths – Semi-pyramids

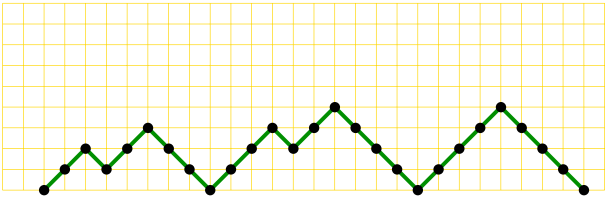


Bijection: Dyck paths – Semi-pyramids



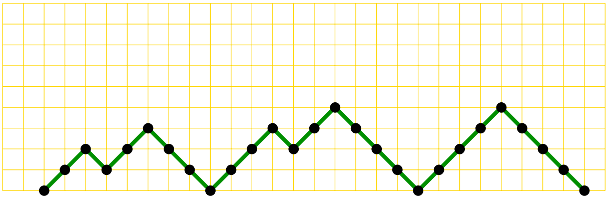
Exercise: Find the reciprocal bijection.

The height of Dyck paths



The height of a Dyck path is its maximum level.

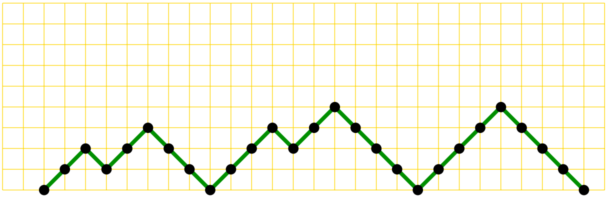
The logarithmic height of Dyck paths



A Dyck path w has the height $h(w)$. Then the logarithmic height $\ell h(w)$ is

$$\lfloor \log_2(1 + h(w)) \rfloor$$

The logarithmic height of Dyck paths



$$\ell h(w) = k$$

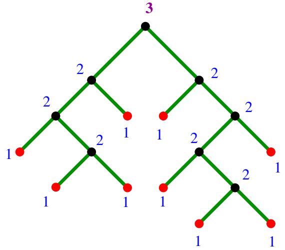
$$\Leftrightarrow 2^k - 1 \leq h(w)) \leq 2^{k+1} - 1$$

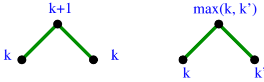
The logarithmic height of Dyck paths

Theorem

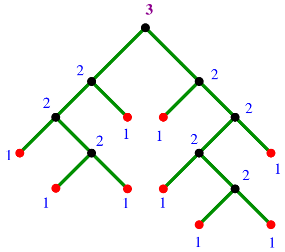
The number of Dyck paths of length $2n$ with logarithmic height k = The number of full binary trees on n internal vertices and with Strahler number k.

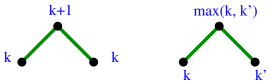
Strahler number





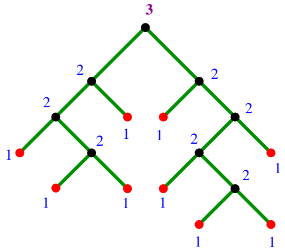
Strahler number

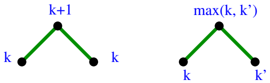




$$S(t,x) = \sum_{n,k} S_{n,k} x^k t^n$$

Strahler number

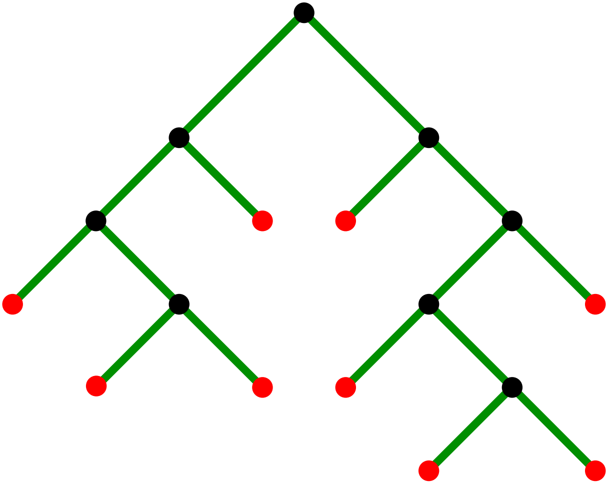




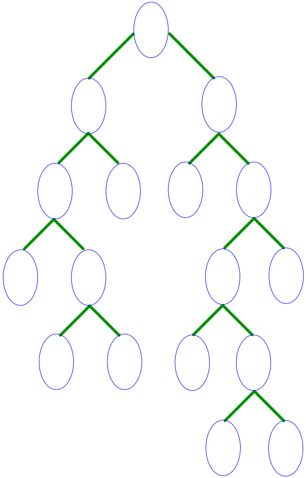
Frangon (1984)
Knuth (2005)

$$S(t,x) = 1 + \frac{xt}{1-2t} S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

Bijection increasing the Strahler number

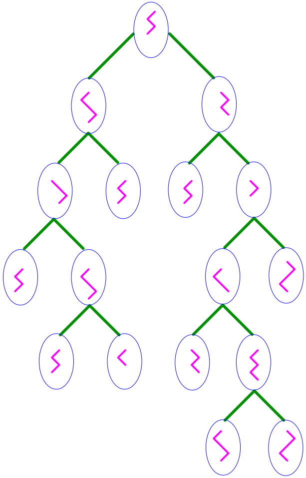


Bijection increasing the Strahler number



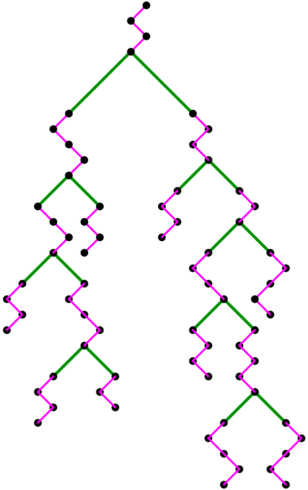
Replace each vertex by a zigzag line.

Bijection increasing the Strahler number

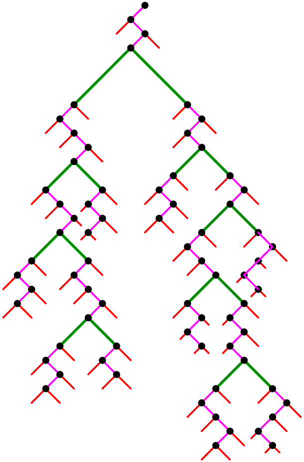


Replace each vertex by a zigzag line.

Bijection increasing the Strahler number



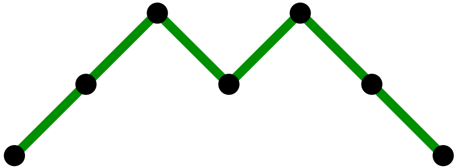
Bijection increasing the Strahler number



Branching out along the zigzag lines.

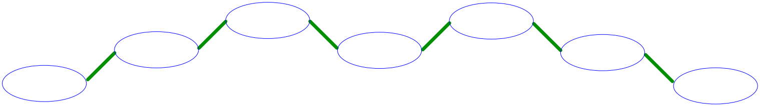
We can do the same with Dyck paths

Bijection increasing the logarithmic height



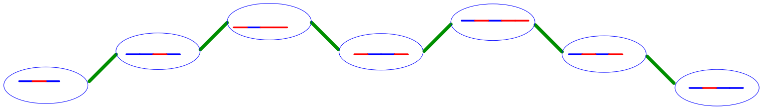
Replace each vertex by a 2-colored horizontal path.

Bijection increasing the logarithmic height



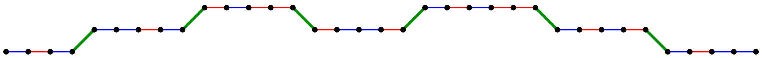
Replace each vertex by a 2-colored horizontal path.

Bijection increasing the logarithmic height



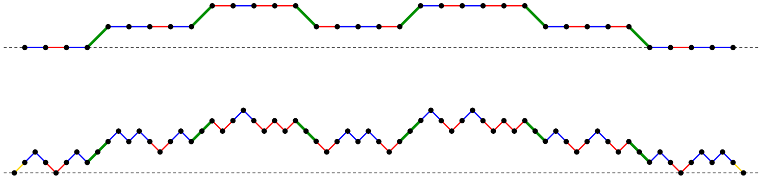
Replace each vertex by a 2-colored horizontal path.

Bijection increasing the logarithmic height



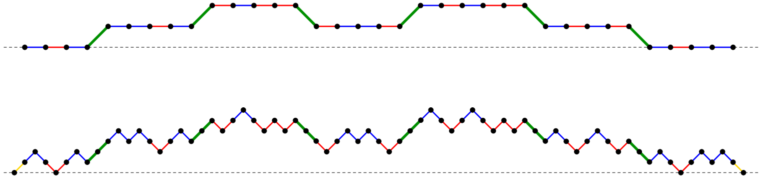
Obtaining a 2-colored Motzkin path.

Bijection increasing the logarithmic height



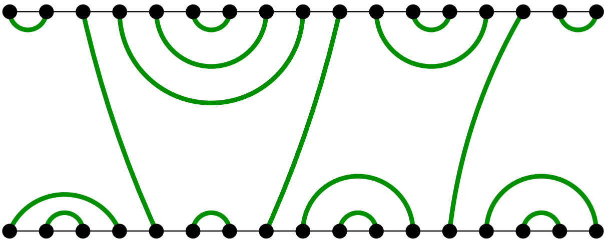
Converting back a Dyck path.

Bijection increasing the logarithmic height

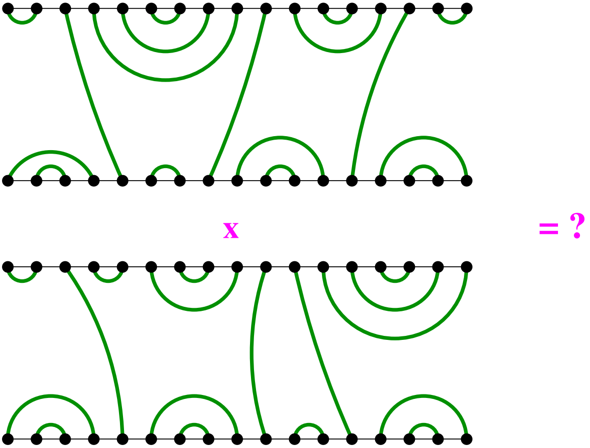


Converting back a Dyck path.
Exercise: Prove that the process increases the logarithmic height 1 unit.

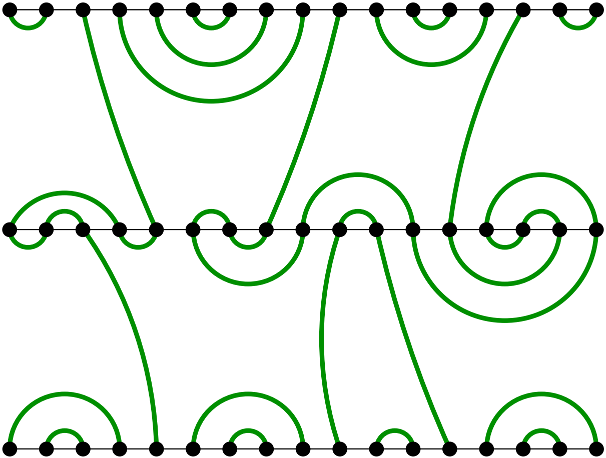
(Another) Chord diagram



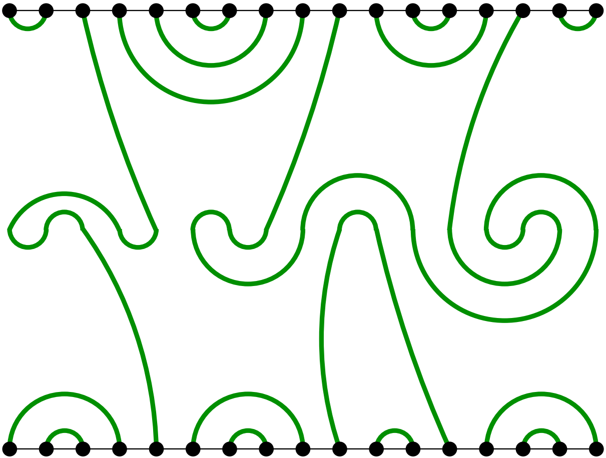
Multiplying Two Chord Diagrams



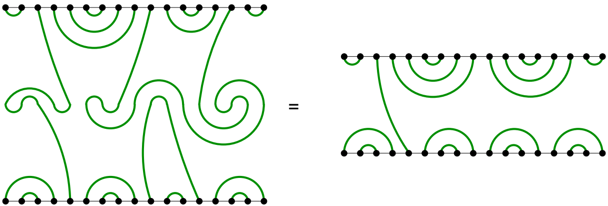
Multiplying Two Chord Diagrams



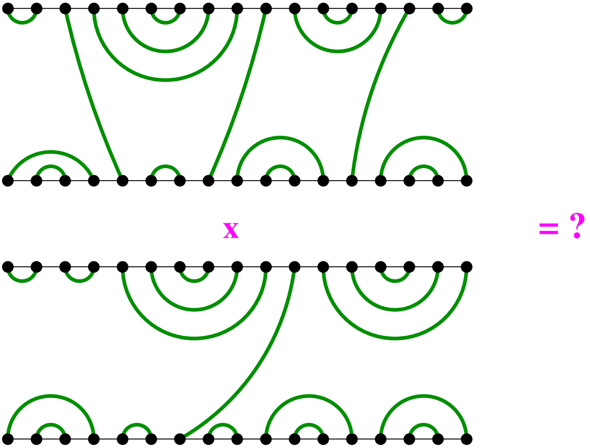
Multiplying Two Chord Diagrams



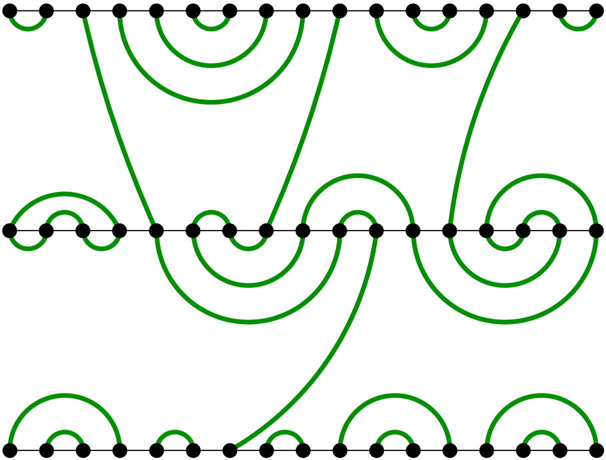
Multiplying Two Chord Diagrams



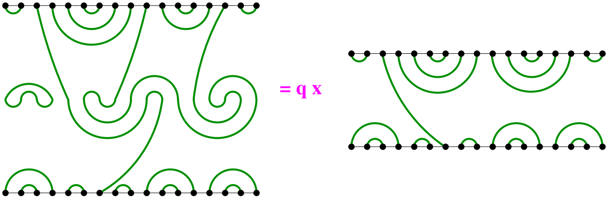
Multiplying Two Chord Diagrams



Multiplying Two Chord Diagrams



Multiplying Two Chord Diagrams



Temperley - Lieb Algebra

The Temperley-Lieb algebra $TL_n(q)$ generated by $\{e_1, e_2, \ldots, e_{n-1}\}$:

1. $e_i^2 = qe_i$
2. $e_ie_j = e_je_i$ if $|i - j| > 1$
3. $e_ie_{i+1}e_i = e_ie_{i-1}e_i = e_i$

Diagram of e_i

e_i =

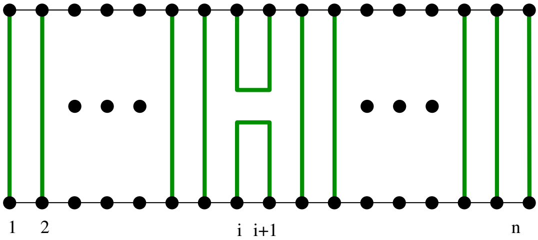
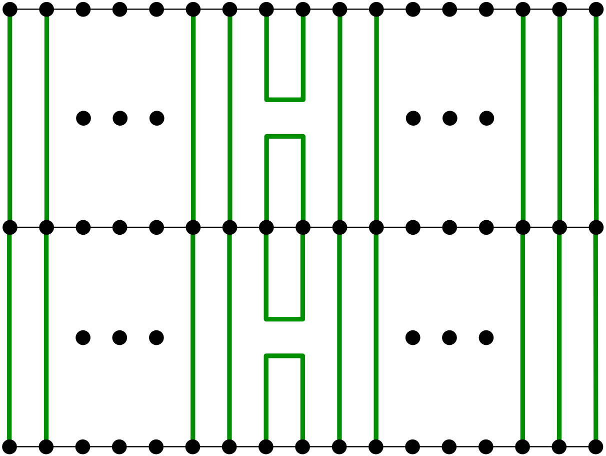
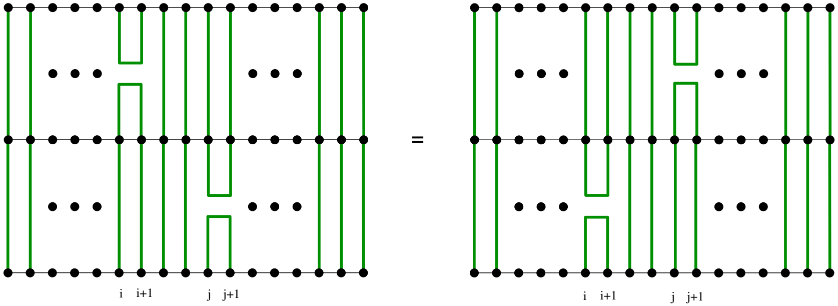


Diagram of e_i



$$e_i^2 = q\, e_i$$

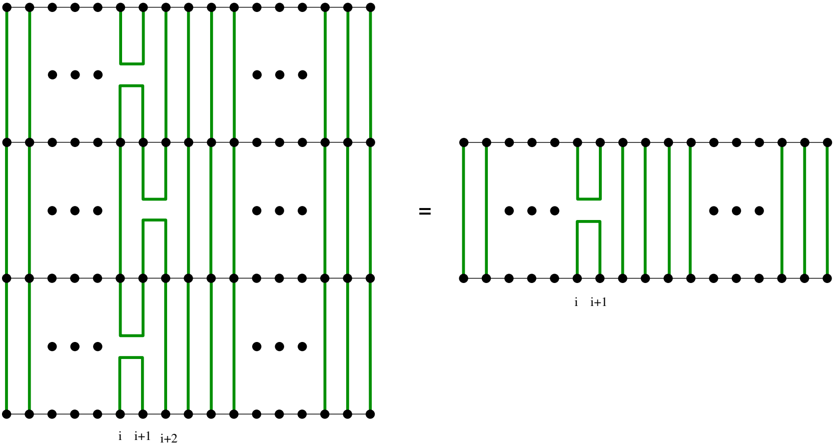
Diagram of e_i



$$e_i e_j = e_j e_i$$

$|i-j| > 1$

Diagram of e_i

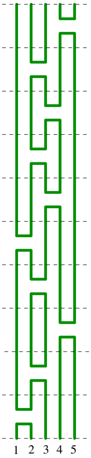


$e_i e_{i+1} e_i = e_i$

Exercise

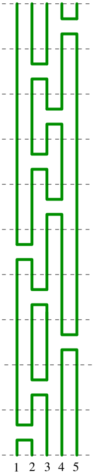
Simplify: $e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_3 e_2 e_4$

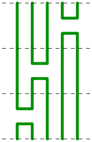
Exercise



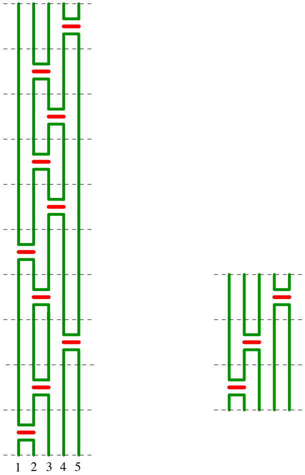
$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_3 e_2 e_4$

Exercise





$$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_3 e_2 e_4 = q\, e_1 e_2 e_4$$

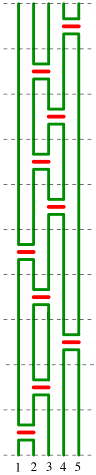


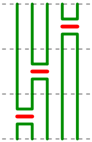
$$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_3 e_2 e_4 = \mathbf{q}\, e_1 e_2 e_4$$

The Normalized Temperley-Lieb algebra $NTL_n(q)$ generated by $\{e_1, e_2, \ldots, e_{n-1}\}$ is a Temperley-Lieb algebra with the following patterns forbidden

1. e_i^2
2. $e_i e_{i+1} e_i$
3. $e_i e_{i-1} e_i$

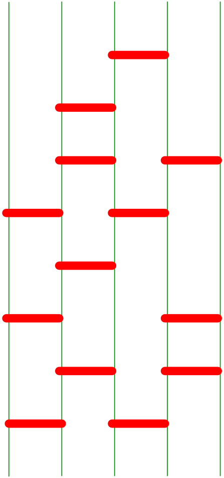
Temperley -Lieb Algebra and Heap



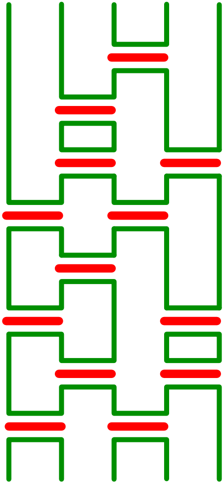


$$e_1 e_2 e_4 e_2 e_1 e_3 e_2 e_3 e_2 e_4 = q\, e_1 e_2 e_4$$

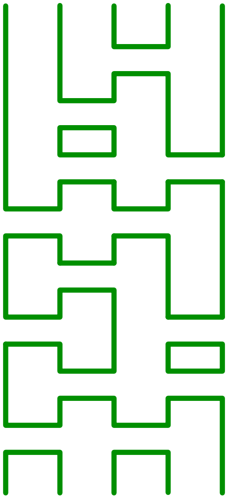
Temperley -Lieb Algebra and Heap



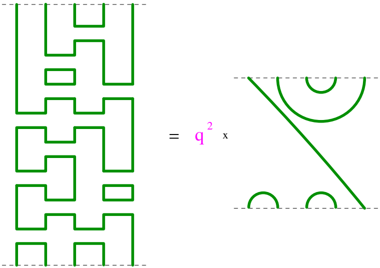
Temperley -Lieb Algebra and Heap

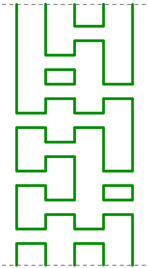


Temperley -Lieb Algebra and Heap

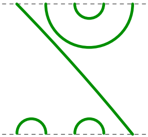


Temperley -Lieb Algebra and Heap

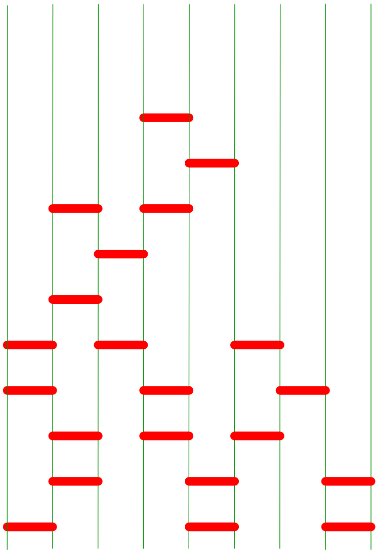




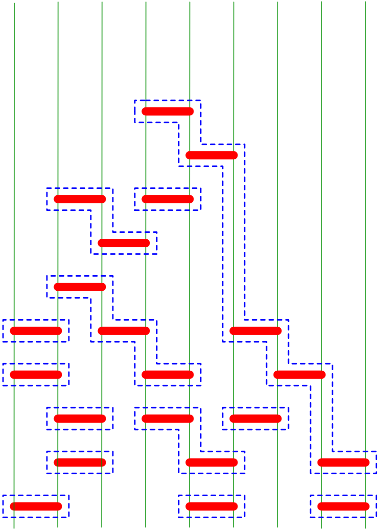
$= q^2 \times$



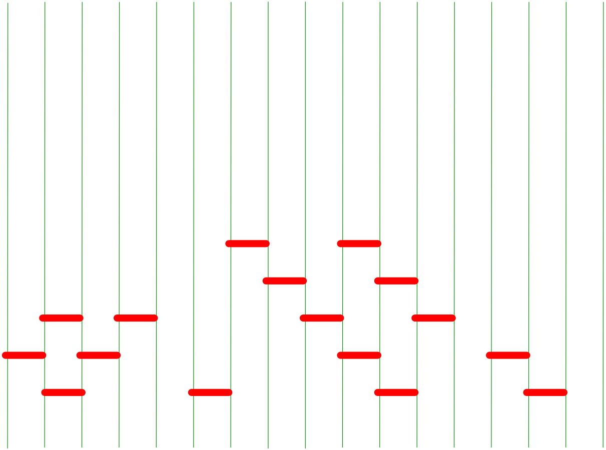
Staircase Decomposition



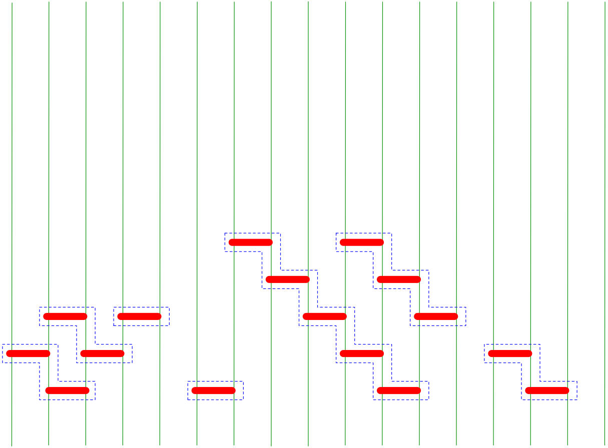
Staircase Decomposition



Staircase Decomposition (Strict Heap)



Staircase Decomposition (Strict Heap)



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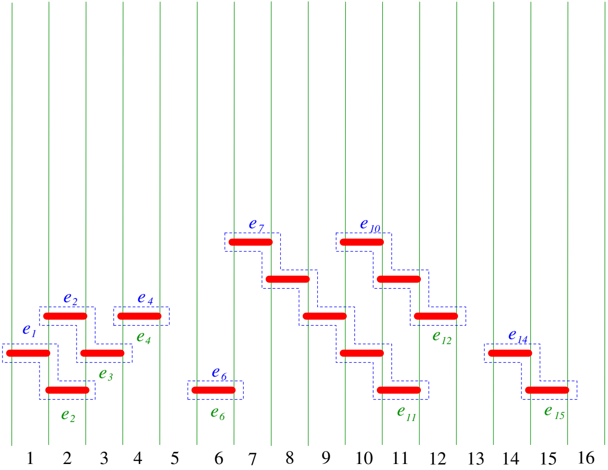
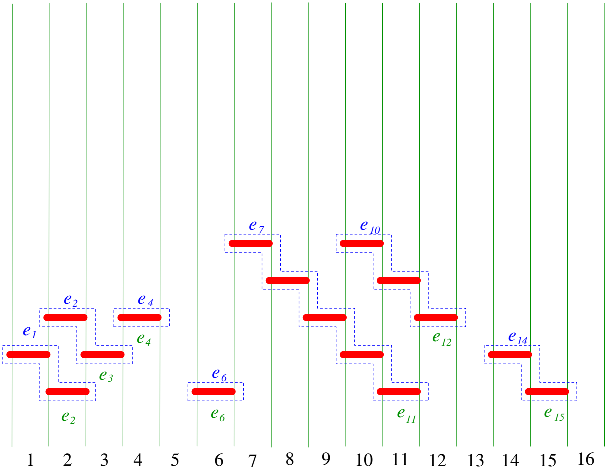


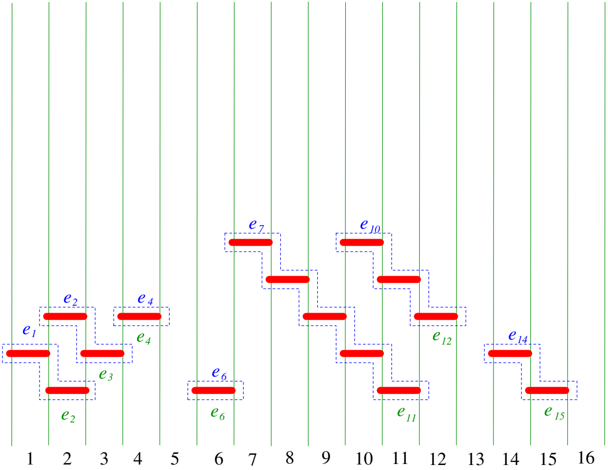
Staircase Decomposition (Strict Heap)



Lemma

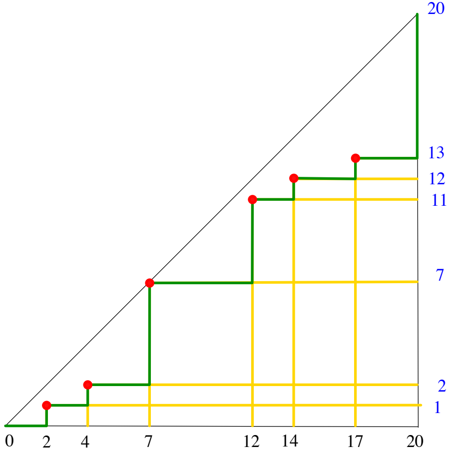
The the higher staircase has the top (bottom) dimer strictly to the left of that of the lower staircase.

Staircase Decomposition (Strict Heap)

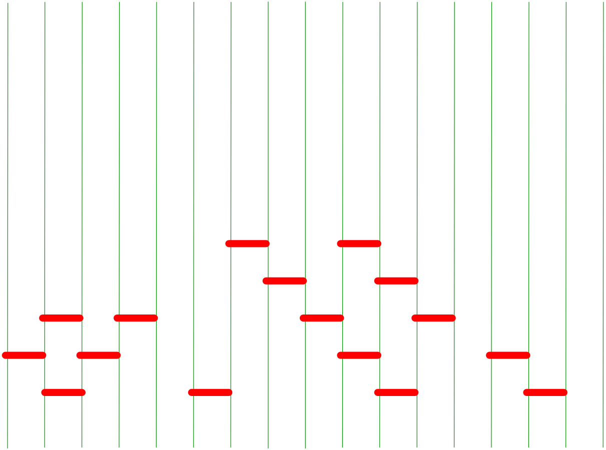


1	2	4	6	7	10	14
2	3	4	6	11	12	15

Recall: Bijection: Ordered Pairs –Dyck Paths



Strict Heaps – Staircase Polygons



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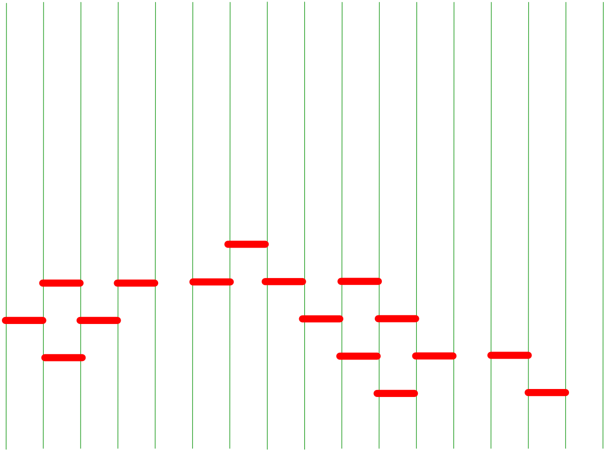

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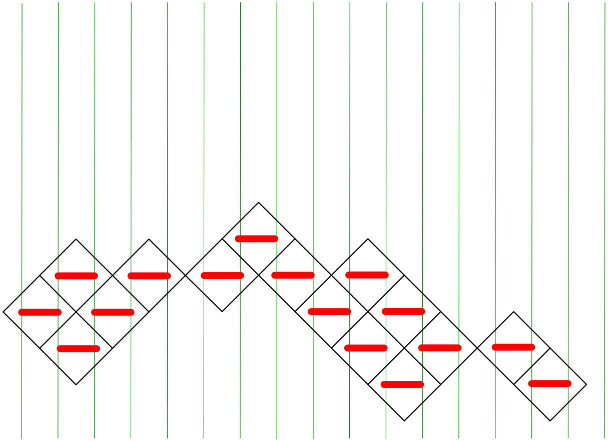

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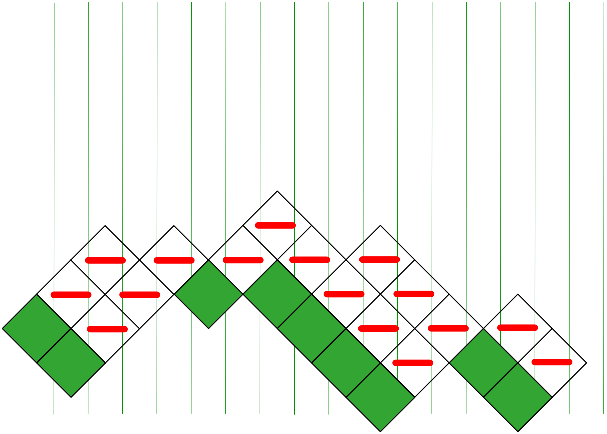
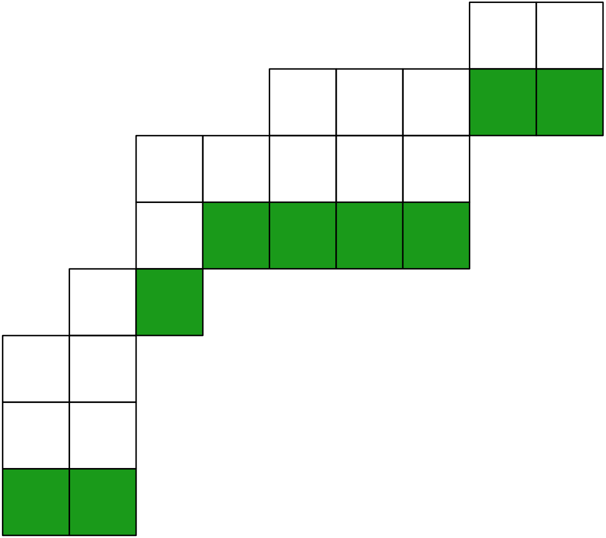
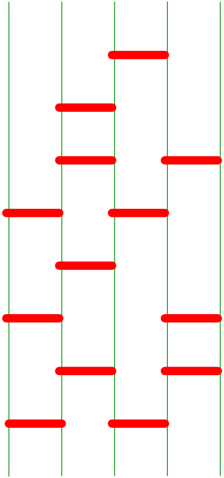


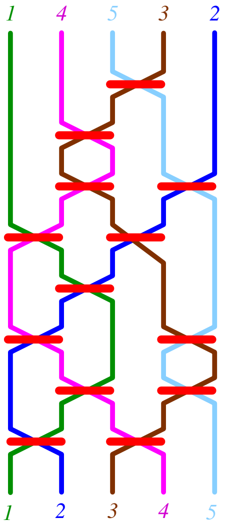
Strict Heaps – Staircase Polygons



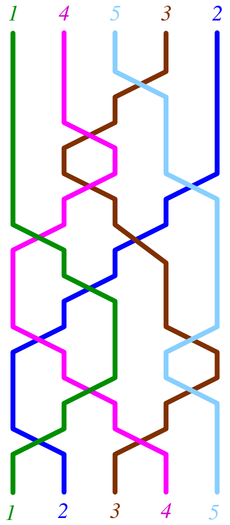
Heaps – Permutations



Heaps – Permutations



Heaps – Permutations



A permutation σ is 321-avoiding if there no $i < j < k$ such that $\sigma(i) > \sigma(j) > \sigma(k)$.

Theorem

Strict heaps are equinumerous to 321-avoiding permutations.

Presentation Topic 1: Permutations with forbidden patterns and Catalan numbers.

q-analogs of Catalan Numbers

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! := [1]_q \cdot [2]_q \cdot [3]_q \ldots [n]_q$
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$

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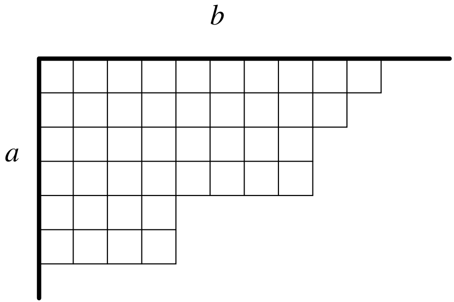
q-binomial Coefficients

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q = ?$$

q-analogs of Catalan Numbers



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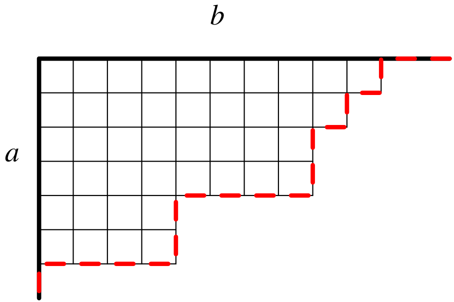
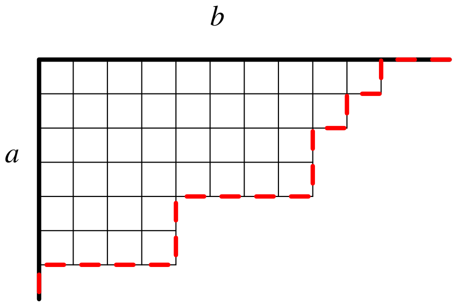
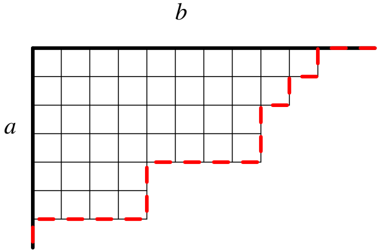


$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q = ?$$



$$\sum_{\mathcal{F}\subset[a\times b]} q^{area(\mathcal{F})} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

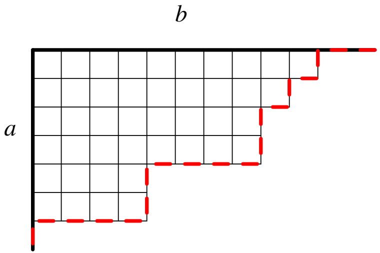
q-analogs of Catalan Numbers



$$\sum_{\mathcal{F} \subset [a \times b]} q^{area(\mathcal{F})} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

Exercise: Prove that the number of k-dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

q-analogs of Catalan Numbers



$$\sum_{\mathcal{F}\subset[a\times b]} q^{area(\mathcal{F})} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

Exercise: Denote by $p(j,k,n)$ the number of integer partitions of n into at most k parts and each part is at most j. Then

$$\sum_n p(j,k,n) q^n = \begin{bmatrix} j+k \\ k \end{bmatrix}_q .$$

A q-analog of Catalan numbers

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

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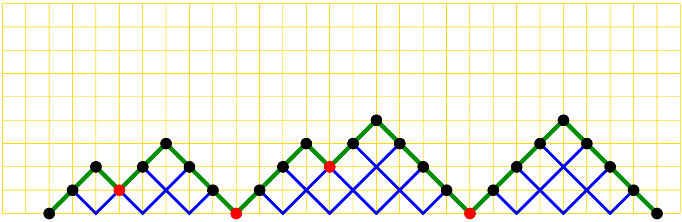
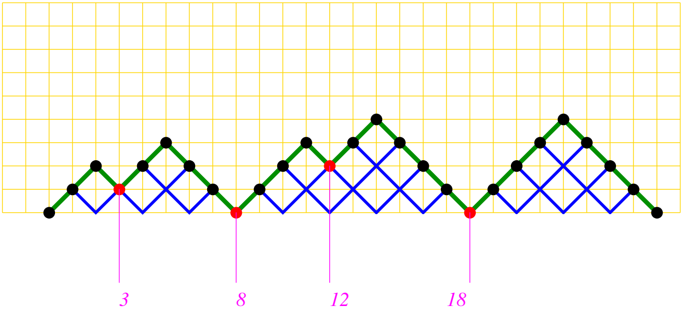


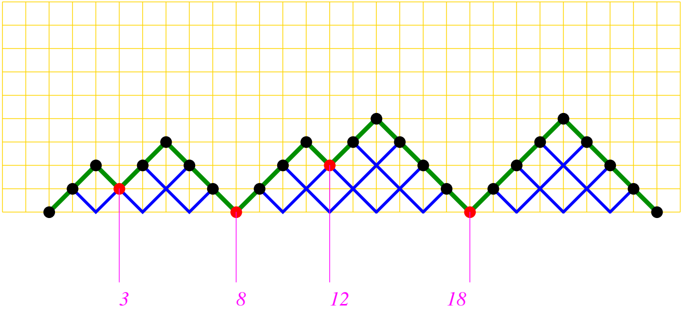
$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

q-analogs of Catalan Numbers



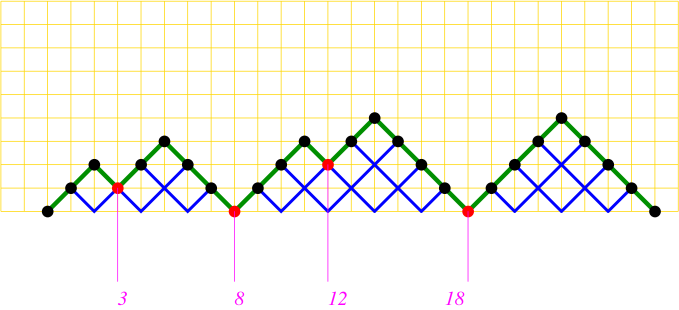
$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

$maj(w) = 3 + 8 + 12 + 18 = 31$



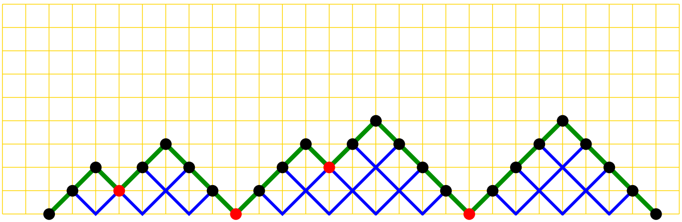
$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

q-analogs of Catalan Numbers



$$\sum_{Dyck\ path\ w;\ |w|=2n} q^{maj(w)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

$area(w) = 18$



q-analogs of Catalan Numbers

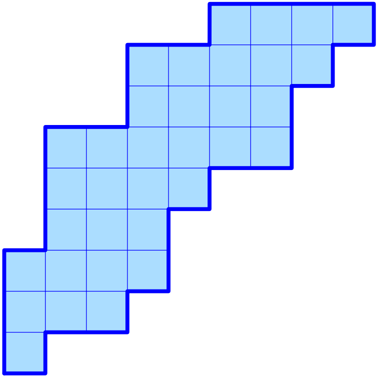
$$\sum_{Dyck\ \ path\ \ w} q^{area(w)} t^{|w|/2}$$

q-analogs of Catalan Numbers

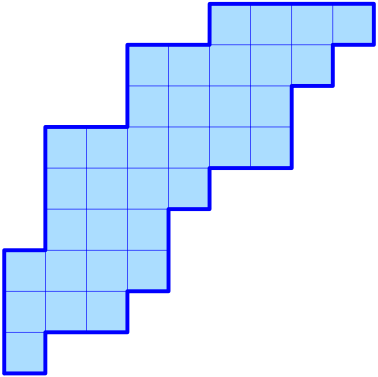
Theorem

$$\sum_{\text{Dyck path } w} q^{area(w)} t^{|w|/2} = \cfrac{1}{1-\cfrac{t}{1-\cfrac{tq}{1-\cfrac{tq^2}{1-\cfrac{tq^3}{1-\cfrac{tq^4}{1-\cfrac{tq^5}{1-\cfrac{tq^6}{1-\cdots}}}}}}}}$$

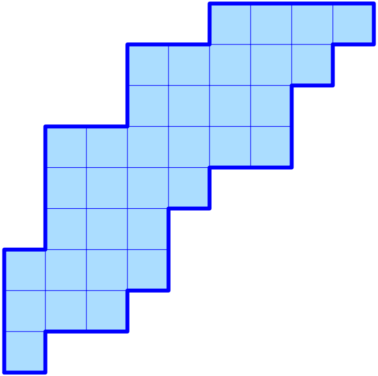
Polya q-Catalan Numbers



Polya q-Catalan Numbers

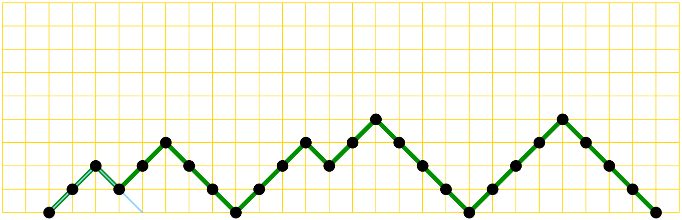


Polya q-Catalan Numbers

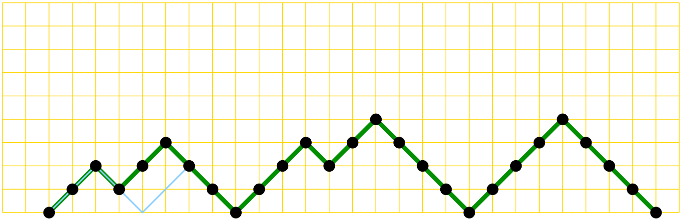


q-Bessel functions

Bounce



Bounce



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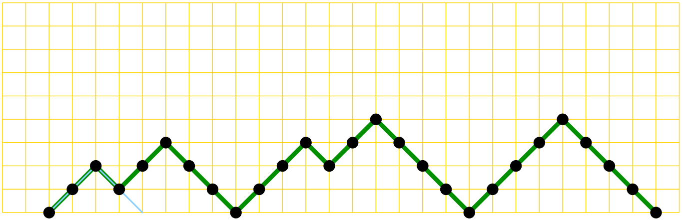

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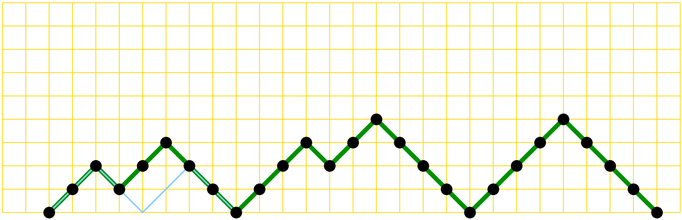
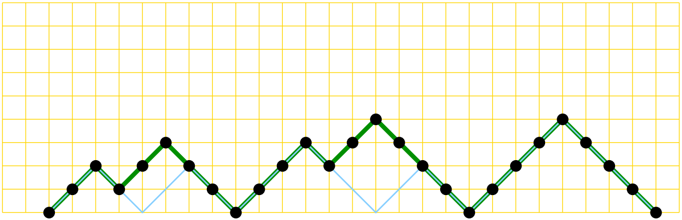
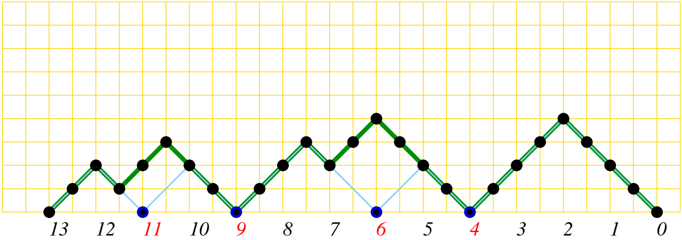


Bounce



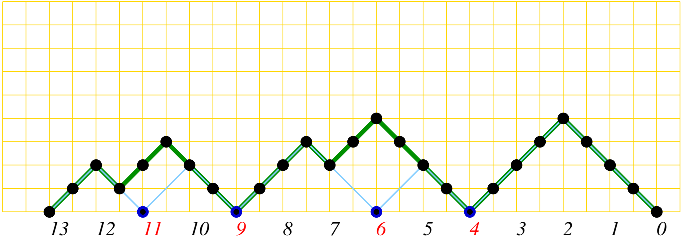
(q,t)-Catalan Numbers

$bounce(w) = 4 + 6 + 9 + 11$



(q, t)-Catalan Numbers

$bounce(w) = 4 + 6 + 9 + 11 = 30$



(q, t)-Catalan Numbers

area and bounce statistics have the same distribution!

(q,t)-Catalan Numbers

area and bounce statistics have the same distribution!

$$\sum_{Dyck\ \ path\ \ w} q^{area(w)} = \sum_{Dyck\ \ path\ \ w} q^{bounce(w)}$$

(q, t)-Catalan Numbers

(q, t)-Catalan Numbers:

$$C_n(q,t) = \sum_{Dyck\ \ path\ \ w;\ \ |w|=2n} q^{area(w)} t^{bounce(w)}$$

(q, t)-Catalan Numbers

$$C_n(q,t) = \sum_{Dyck\ \ path\ \ w;\ \ |w|=2n} q^{area(w)} t^{bounce(w)}$$

This polynomial is symmetric in q, t:

$$C_n(q,t) = C_n(t,q)$$

(q,t)-Catalan Numbers

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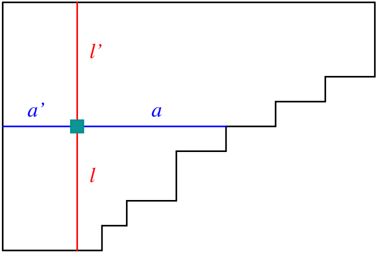
This polynomial is symmetric in q, t:

$$C_n(q,t) = C_n(t,q)$$

There is no bijective proof!

Original definition of (q,t)-Catalan Numbers



A. Garsia, M. Haiman (1994)

$$C_n(q,t) = \sum_{\lambda \vdash n} \frac{t^{2\sum_{c\in\lambda} l} q^{2\sum_{c\in\lambda} a}(1-t)(1-q)\prod_{c\in\lambda}(1-q^{a'}t^{l'})\sum_{c\in\lambda} q^{a'}t^{l'}}{\prod_{c\in\lambda}(q^a - t^{l+1})(t^l - q^{a+1})}$$

Presentation Topic 2: (q,t)-Catalan numbers.
Presentation Topic 3: 'Kepler Towers' and Catalan numbers.